Transport and Communications Science Journal

EXISTENCE AND UNIQUENESS OF SOLUTIONS FOR TWO-DIMENSIONAL FRACTIONAL NON- COLLIDING PARTICLE SYSTEMS

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ARTICLE INFO

TYPE: Research Article
Received: 5/11/2019
Revised: 2/12/2019
Accepted: 5/12/2019
Published online: 31/1/2020
https://doi.org/10.25073/tcsj.71.1.2

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Abstract. In this paper, we consider the stochastic evolution of two particles with electrostatic repulsion and restoring force which is modeled by a system of stochastic differential equations driven by fractional Brownian motion where the diffusion coefficients are constant. This is the simplest case for some classes of non- colliding particle systems such as Dyson Brownian motions, Brownian particles systems with nearest neighbour repulsion. We will prove that the equation has a unique non- colliding solution in path- wise sense.

Keywords: stochastic differential equation, fractional Brownian motion, non- colliding particle systems.

1. INTRODUCTION

It is known that the systems of SDEs driven by *standard Brownian motion* describing positions of d ordered particles evolving in R has the form

$$dx_i(t) = \left\{ \sum_{j\neq i} \frac{\gamma_{ij}}{x_i(t) - x_j(t)} + b_i(t, x(t)) \right\} dt + \sum_{j=1}^{m} \sigma_{ij}(x(t)) dW_j(t), i = 1,...d, \quad (1)$$

where $W = (W_1(t), W_2(t), ..., W_m(t))$ is a m - dimensional standard Brownian. The system of SDEs (2) is a type of SDEs whose solution stays in a domain which has been studied by many authors because of its important applications in physics, biology and finance [1]. In mathematical physics, the process $x(t)$ is used to model systems of d non-colliding particles with electrostatic repulsion and restoring force. It contains Dyson Brownian Motions, Squared Bessel particle systems, Jacobi particle systems, non-colliding Brownian and Squared Bessel particles, potential-interacting Brownian particles and other particle systems crucial in mathematical physics and physical statistics [2, 3]. The existence and uniqueness of a strong non-colliding solution to such kind of systems have been intensively studied by many authors ([4, 5, 6, 7] and the references therein). *But there are no results in the case of fractional non-colliding particles.*

The main aim of this paper is to study the two- dimensional *fractional* non-colliding particle systems

$$\begin{cases} dX_1(t) = \left(\dfrac{\gamma}{X_1(t) - X_2(t)} + b_1(t, X(t)) \right) dt + \sum_{j=1}^{m} \sigma_{1j} dB_j^H(t), \\ dX_2(t) = \left(\dfrac{\gamma}{X_2(t) - X_1(t)} + b_2(t, X(t)) \right) dt + \sum_{j=1}^{m} \sigma_{2j} dB_j^H(t). \end{cases} \quad (2)$$

where $X(0) = (X_1(0), X_2(0)) \in \Delta_2 = \{x = (x_1, x_2)^T \in \mathbb{R}^2 : x_1 < x_2\}$ almost surely (a.s) and $B = \{B^H(t), t \ge 0\} = (B_1^H(t), B_2^H(t), ..., B_m^H(t))^T$ is an m-dimensional *fractional Brownian motion* with the Hurst parameter $H \in (\frac{1}{2}, 1)$ defined on a complete probability space $(\Omega, \mathcal{F}, P)$ with a filtration $\{\mathcal{F}_t, t \ge 0\}$ satisfying the usual conditions. We prove that equation (1) has a unique non- colliding solution in path-wise sense. To the best of my knowledge, this is the first paper to discuss the fractional non- colliding particle systems.

2. THE EXISTENCE AND UNIQUENESS OF THE SOLUTION

Fix $T > 0$ and we consider eq. (1) on the interval $[0, T]$. We suppose that the coefficients $b_i : [0; +\infty) \times \mathbb{R}^2 \to \mathbb{R}$ are measurable functions and there exist positive constants L, C such that following conditions hold

(i) $X(0) \in \Delta_2$ almost surely.

(ii) $\gamma > 0$.

(iii) b_i(t,x), $i = 1, 2$ are globally Lipschitz continuous with respect to x, that is

$$\sup_{i=1,2} |b_i(t,x) - b_i(t,y)| \le L|x - y|,$$

for all $x, y \in \mathbb{R}^2$ and $t \in [0, T]$.

(iv) $b_i(t, x)$, $i = 1, 2$ are sub-linearly growth with respect to x, that is

$$\sup_{i=1,2} |b_i(t,x)| \le C(1 + |x|),$$

for all $x \in \mathbb{R}^2$ and $t \in [0, T]$.

(v) $b_1(t,x) < b_2(t,x)$ for all $x \in \mathbb{R}^2$ and $t \in [0, T]$.

Denote $a \vee b = \max\{a,b\}$ and $a \wedge b = \min\{a,b\}$. For each $n \in \mathbb{N}$, we consider the following fractional SDEs

$$\begin{cases} dX_1^n(t) = \left(\dfrac{\gamma}{(X_1^n(t) - X_2^n(t)) \wedge \frac{-1}{n}} + b_1(t, X^n(t)) \right) dt + \sum_{j=1}^{m} \sigma_{1j} dB_j^H(t), \\ dX_2^n(t) = \left(\dfrac{\gamma}{(X_2^n(t) - X_1^n(t)) \vee \frac{1}{n}} + b_2(t, X^n(t)) \right) dt + \sum_{j=1}^{m} \sigma_{2j} dB_j^H(t), \end{cases} \tag{3}$$

where $X^n(0) = (X_1^n(0), X_2^n(0)) \in \Delta_2$. For each $n \in \mathbb{N}$ and $x = (x_1, x_2)$ we set

$$f_1^n(t,x) = \frac{\gamma}{(x_1 - x_2) \wedge \frac{-1}{n}} + b_1(t,x),$$

$$f_2^n(t,x) = \frac{\gamma}{(x_2 - x_1) \vee \frac{1}{n}} + b_2(t,x).$$

Lemma 2.1. *For each $T > 0$, eq. (3) has a unique solution on $[0,T]$.*

Proof: Using the estimate $|a \vee c - b \vee c| \le |a-b|, |a \wedge c - b \wedge c| \le |a-b|,$ it is straightforward to verify that

$$\left| f_i^n(t,x) - f_i^n(t,y) \right| \le (\sqrt{2}\gamma n^2 + L)|x-y|,$$

for all $x = (x_1, x_2)$ and $t \in [0,T]$ and

$$\left| f_i^n(t,x) \right| \le n\gamma + C(1 + |x|).$$

It means that coefficients of eq. (3) satisfy Lipschitz continuity and boundedness condition. Hence it follows from Theorem 2.1 in [8] that eq. (3) has a unique solution on the interval $[0,T]$.

We recall a result on the modulus of continuity of trajectories of fractional Brownian motion ([9])

Lemma 2.2. *Let $B = \{B^H(t), t \ge 0\}$ be a fractional Brownian motion of Hurst parameter $H \in (0,1)$. Then for every $0 < \varepsilon < H$ and $T > 0$, there exists an event $\Omega_{\varepsilon,T}$ with $P(\Omega_{\varepsilon,T}) = 1$, and a positive random variable $\eta_{\varepsilon,T}$ such that $\mathrm{E}(|\eta_{\varepsilon,T}|^p) < \infty$ for all $p \in [1,\infty)$ and for all $s,t \in [0,T]$,*

$$\left| B^H(t,\omega) - B^H(s,\omega) \right| \le \eta_{\varepsilon,T}(\omega) |t-s|^{H-\varepsilon}, \text{ for any } \omega \in \Omega_{\varepsilon,T}.$$

We denote

$$\tau_n = \inf\{t \in [0,T] : \left| X_2^n(t) - X_2^n(t) \right| \le \tfrac{1}{n}\} \wedge T.$$

In order to prove that eq. (1) has a unique solution on $[0,T]$, we need the following lemma.

Lemma 2.3. *The sequence* τ_n *is non-decreasing, and for almost all* $\omega \in \Omega, \tau_n(\omega) = T$ *for* n *large enough.*

Proof. Using the estimate $-(a \wedge b) = -a \vee -b,$ from eq. (3) we have

$$d(X_2^n(t) - X_1^n(t)) = \left(\frac{2\gamma}{(X_2^n(t) - X_1^n(t)) \vee \frac{1}{n}} + b_2(t, X^n(t)) - b_1(t, X^n(t)) \right) dt + \sum_{j=1}^{m} (\sigma_{2j} - \sigma_{1j}) dB_j^H(t). \tag{4}$$

We set $Y^n(t) = X_2^n(t) - X_1^n(t).$ Eq. (4) becomes

$$d(Y^n(t)) = \left(\frac{2\gamma}{Y^n(t) \vee \frac{1}{n}} + b_2(t, X^n(t)) - b_1(t, X^n(t)) \right) dt + \sum_{j=1}^{m} (\sigma_{2j} - \sigma_{1j}) dB_j^H(t). \tag{5}$$

Then $Y^n(0) > 0$ and $\tau_n = \inf\{t \in [0,T] : \left| Y^n(t) \right| \le \tfrac{1}{n}\} \wedge T.$

It follows from Lemma 2.2 that for any $\varepsilon \in (0, H - \tfrac{1}{2}),$ there exist a finite random variable $\eta_{\varepsilon,T}$ and an event $\Omega_{\varepsilon,T} \in \mathcal{F}$ which do not depend on n such that $\mathrm{P}(\Omega_{\omega,T}) = 1,$ and

$$\left| \sum_{j=1}^{m} (\sigma_{2j} - \sigma_{1j})(B_j^H(t,\omega) - B_j^H(s,\omega) \right| \le \eta_{\varepsilon,T}(\omega) \left| t - s \right|^{H-\varepsilon}, \text{ for any } \omega \in \Omega_{\varepsilon,T} \text{ and } 0 \le s < t \le T. \tag{6}$$

We will adapt the contradiction method in [10]. Assume that for some $\omega_0 \in \Omega_{\varepsilon,T}, \tau_n(\omega_0) < T$ for all $n \in \mathbb{N}.$ By virtue of the continuity of sample paths of $Y^n,$ it follows from the definition of τ_n that $Y^n(\tau_n(\omega_0), \omega_0) = \dfrac{1}{n}$ and $Y^n(t, \omega_0) \ge \dfrac{1}{n}$ for all $t \in [0, \tau_n(\omega_0)].$ Denote

$$\kappa_n(\omega_0) = \sup\{t \in [0, \tau_n(\omega_0)] : Y^n(t, \omega_0) \ge \frac{2}{n}\}.$$

We have

$$\frac{1}{n} \le Y^n(t, \omega_0) \le \frac{2}{n}, \text{ for all } t \in [\kappa_n(\omega_0), \tau_n(\omega_0)].$$

In order to simplify our notations, we will omit ω_0 in brackets in further formulas. We have

$$Y^n(\tau_n) - Y^n(\kappa_n) = -\frac{1}{n} = \int_{\kappa_n}^{\tau_n} \left(\frac{2\gamma}{Y^n(s)} + b_2(s, X^n(s)) - b_1(s, X^n(s)) \right) ds + \sum_{j=1}^{m} (\sigma_{2j} - \sigma_{1j})(B_j^H(\tau_n) - B_j^H(\kappa_n)).$$

This implies

$$\left|\sum_{j=1}^{m}(\sigma_{2j}-\sigma_{ij})(B_j^H(\tau_n)-B_j^H(\kappa_n))\right| = \left|\frac{1}{n}+\int_{\kappa_n}^{\tau_n}(\frac{2\gamma}{Y^n(s)}+b_2(s,X^n(s))-b_1(s,X^n(s))ds\right|. \qquad (7)$$

Note that for all $s\in[\kappa_n,\tau_n]$

$$\frac{2\gamma}{Y^n(s)}+b_2(s,X^n(s))-b_1(s,X^n(s)\geq 4n\gamma.$$

Then for all $n\geq n_0=\frac{2}{Y^n(0)}$, it follows from eq. (7) that

$$\left|\sum_{j=1}^{m}(\sigma_{2j}-\sigma_{ij})(B_j^H(\tau_n)-B_j^H(\kappa_n))\right|\geq\frac{1}{n}+4n\gamma(\tau_n-\kappa_n).$$

This fact together with eq. (6) implies that

$$\eta_{\varepsilon,T}\left|\tau_n-\kappa_n\right|^{H-\varepsilon}\geq\frac{1}{n}+4n\gamma(\tau_n-\kappa_n),\ \text{for all } n\geq n_0 \qquad (8)$$

By following similar arguments in the proof of Theorem 2 in [10], we see that the inequality (8) fails for all *n* large enough. This contradiction completes the proof of the lemma.

We consider the process $\{X(t)=(X_1(t),X_2(t))\}_{t\geq 0}$ which satisfies equation (1). Now, we set $Y(t)=X_2(t)-X_1(t)$, then $Y(t)$ satisfies the following equation

$$d(Y(t))=\left(\frac{2\gamma}{Y(t)}+b_2(t,X(t))-b_1(t,X(t))\right)dt+\sum_{j=1}^{m}(\sigma_{2j}-\sigma_{1j})dB_j^H(t). \qquad (9)$$

Lemma 2.4. *If eq. (1) has a solution then* $Y(t)=X_2(t)-X_1(t)>0$ *for all* $t\in[0,T]$ *almost surely.*

Proof. We will also use the contradiction method. Assume that for some $\omega_0\in\Omega$, $\inf_{t\in[0,T]}Y(t,\omega_0)=0$. Denote $\tau=\inf\{t:Y(t,\omega_0)=0\}$. For each $n\geq 1$ we denote $\nu_n=\sup\{t<\tau:Y(t,\omega_0)=\frac{1}{n}\}$. Since Y has continuous sample paths, $0<\nu_n<\tau\leq T$ and $Y(t,\omega_0)\in(0,\frac{1}{n})$ for all $t\in(\nu_n,\tau)$. We have

$$Y(\tau)-Y(\nu_n)=-\frac{1}{n}=\int_{\nu_n}^{\tau}\left(\frac{2\gamma}{Y(s)}+b_2(s,X(s))-b_1(s,X(s))\right)ds+\sum_{j=1}^{m}(\sigma_{2j}-\sigma_{ij})(B_j^H(\tau)-B_j^H(\nu_n)).$$

Note that for all $s\in[\nu_n,\tau]$

$$\frac{2\gamma}{Y(s)}+b_2(s,X(s))-b_1(s,X(s))\geq 2n\gamma.$$

So we have

$$\left|\sum_{j=1}^{m}(\sigma_{2j}-\sigma_{ij})(B_j^H(\tau)-B_j^H(\nu_n))\right| \geq \frac{1}{n}+2n\gamma(\tau-\nu_n). \quad (10)$$

Again using the inequality (6), we have

$$\eta_{\varepsilon,T}\left|\tau-\nu_n\right|^{H-\varepsilon} \geq \frac{1}{n}+2n\gamma(\tau-\nu_n). \quad (11)$$

Similar to the argument of Theorem 2 in [10] we see that the inequality (11) fails for all *n* large enough. This contradiction completes the lemma.

Based on above lemmas we obtain the main theorem of this paper which is stated as follows

Theorem 2.5. *For each* $T>0$ *eq. (1) has a unique solution on* $[0,\ T]$.

Proof. First, from Lemma 2.3, there exists a finite random variable n_0 such that $X_2^n(t)-X_2^n(t)\geq \frac{1}{n_0}>0$ almost surely for any $t\in[0,T]$. Therefore, the process $X^n(t)=(X_2^n(t),X_2^n(t))$ converges almost surely to a limit, called $X(t)$ when *n* tends to infinity and $X(t)$ satisfies eq. (1). This fact together with Lemma (2.4) leads to eq. (1) has a strong non- colliding solution.

Next, we show that eq. (1) has a unique solution in path-wise sense. Let $X(t)$ and $\overline{X}(t)$ be two solutions of eq. (1) on $[0,T]$. We have

$$\left|X_1(t,\omega)-\overline{X_1}(t,\omega)\right|=$$

$$=\left|\int_0^t\left(\frac{\gamma}{X_1(s,\omega)-X_2(s,\omega)}+b_1(s,X(s,\omega))-\frac{\gamma}{\overline{X_1}(s,\omega)-\overline{X_2}(s,\omega)}-b_1(s,\overline{X}(s,\omega))\right)ds\right|$$

$$\leq\int_0^t\left|\left(\frac{\gamma}{X_1(s,\omega)-X_2(s,\omega)}-\frac{\gamma}{\overline{X_1}(s,\omega)-\overline{X_2}(s,\omega)}\right)\right|ds+\int_0^t\left|b_1(s,X(s,\omega))-b_1(s,\overline{X}(s,\omega))\right|ds. \quad (12)$$

Using the continuous property of the sample paths of $X(t)$ and $\overline{X}(t)$ and Lemma 2.4, we have

$$m_0=\min_{t\in[0,T]}\{X_2(t,\omega)-X_1(t,\omega),\overline{X}_2(t,\omega)-\overline{X}_1(t,\omega)\}>0.$$

This fact together with the Lipschitz condition of *b* leads to

$$\left|X_1(t,\omega)-\overline{X_1}(t,\omega)\right|\leq\int_0^t\frac{\gamma\left|(X_2(s,\omega)-\overline{X}_2(s,\omega))-(X_1(s,\omega)-\overline{X}_1(s,\omega))\right|}{m_0^2}+\int_0^t L\left|X(s,\omega)-\overline{X}(s,\omega)\right|ds. \quad (13)$$

Similarly, we estimate $\left|X_2(t,\omega)-\overline{X_2}(t,\omega)\right|$. We obtain

$$\sum_{i=1}^{2}\left|X_i(t,\omega)-\overline{X_i}(t,\omega)\right| \le \left(\frac{2\gamma}{m_0^2}+2L\right)\int_0^t \sum_{i=1}^{2}\left|X_i(s,\omega)-\overline{X_i}(s,\omega)\right| ds. \quad (14)$$

It follows from Gronwall's inequality that

$$\sum_{i=1}^{2}\left|X_i(t,\omega)-\overline{X_i}(t,\omega)\right| = 0, \quad \text{for all } t\in[0,T].$$

Therefore, $X(t,\omega)=\overline{X}(t,\omega)$ for all $t\in[0,T]$. The uniqueness has been concluded.

3. CONCLUSION

The main result proved in this paper is the existence and uniqueness of strong non-colliding solution in path- wise sense to the two- dimensional *fractional* non- colliding particle systems. From this result, we can propose a numerical approximation for this system.

REFERENCES

[1] P. Kloeden, E. Platen, Numerical solution of stochastic differential equations, Springer–Verlag,1995.

[2] M. Katori, H. Tanemura, Noncolliding Squared Bessel processes, J. Stat. Phys., 142 (2011) 592-615. https://doi.org/10.1007/s10955-011-0117-y

[3] M. Katori, H. Tanemura, Noncolliding processes, matrix-valued processes and determinantal processes, Sugaku Expositions, 24 (2011) 263-289. https://doi.org/10.11429/sugaku.0613225

[4] E. Cepa, D. Lepingle, Diffusing particles with electrostatic repulsion, Probab.Theory Related Fields, 107 (1997) 429-449. https://doi.org/10.1007/s004400050092

[5] P. Graczyk, J. Ma lecki, Strong solutions of non-colliding particle systems, Electron. J. Probab, 19(2014) 1-21.

[6] L. C. G. Rogers, Z. Shi, Interacting Brownian particles and the Wigner law, Probab. Theory Related Fields, 95 (1993) 555-570. https://doi.org/10.1007/BF01196734

[7] N. Naganuma, D. Taguchi, Malliavin Calculus for Non-colliding Particle Systems, Stochastic Processes and their Applications, 2019. https://doi.org/10.1016/j.spa.2019.07.005

[8] D. Nualart, A. Rascanu, Differential equations driven by fractional Brownian motion, Collectanea Mathematica, 53 (2002) 177-193.

[9] Y. S. Mishura, Stochastic Calculus for Fractional Brownian Motion and Related Processes, Lecture Notes in Math, Springer, Berlin, 2008.

[10] Y.. Mishura, A. Yurchenko-Tytarenko, Fractional Cox-Ingersoll-Ross process with non-zero "mean", Modern Stochastic: Theory and Applications, 5 (2018) 99-111. https://doi.org/10.15559/18-VMSTA97