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A NEW APPROACH TO EQUILIBRIUM PROBLEMS VIA REGULARIZED EXTRAGRADIENT-TYPE DYNAMICS

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Abstract. Nonlinear analysis serves as a fundamental framework for modeling diverse phenomena in modern science and engineering. Within this field, equilibrium problems provide a powerful tool that not only unifies diverse mathematical models, such as variational inequalities and fixed point problems, but also effectively characterizes various phenomena in various scientific disciplines. Over the past decades, dynamical systems have emerged as an effective method for finding solutions to these problems due to their continuous-time nature and stability properties. This paper investigates a novel regularized dynamical system of extragradient-type specifically designed to solve monotone equilibrium problems. The primary challenge addressed is ensuring the existence of a global solution and achieving strong convergence under mild assumptions.

Our methodological approach involves constructing a continuous-time trajectory governed by a regularization term and an extragradient-type structure. We first prove the existence and uniqueness of a strong global solution to the proposed system. Subsequently, the asymptotic behavior of the system is analyzed, demonstrating the strong convergence of the generated trajectories toward a specific solution of the original equilibrium problem. Lastly, a time discretization scheme is implemented, leading to a new iterative regularized extragradient-type algorithm enhanced by relaxation parameters.

The main results confirm that the proposed dynamical approach not only guarantees theoretical stability but also provides a robust computational platform for monotone equilibrium problems. The findings of this study offer significant insights into the development of efficient continuous-time optimization methods and contribute to the broader understanding of regularization techniques in nonlinear analysis.

Keywords: equilibrium problem, dynamical system, iterative method, regularization.

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1. INTRODUCTION

Throughout this work, let C be a nonempty, closed, and convex subset of a Hilbert space H. Let $f: H \times H \to \mathbb{R}$ be an equilibrium bifunction on C, i.e., $f(x,x) = 0$ for all $x \in C$. We aim to address the following equilibrium problem:

$$\text{Find } x^* \in C \text{ such that } f(x^*, y) \geq 0 \text{ for every } y \in C. \qquad \big(\text{EP}(f,C)\big)$$

The solution set of $\text{EP}(f,C)$ is denoted by $Sol(f,C)$, which is assumed to be nonempty in the sequel.

Equilibrium problems (EPs) have established themselves as a cornerstone of nonlinear analysis, providing a unified framework for various mathematical models, including variational inequalities, fixed-point problems, and Nash equilibria in game theory [1-3]. Due to their extensive applications in economics, engineering, and the physical sciences, the search for efficient numerical solutions has attracted significant research interest. Early developments focused on the proximal point method [4] and the extragradient method [5]. While these foundational approaches are effective, they often only guarantee weak convergence. To achieve strong convergence, which is more desirable for practical computation in infinite-dimensional Hilbert spaces, researchers have integrated additional techniques such as viscosity methods and Tikhonov-type regularization [6, 7].

Recently, the continuous-time dynamical system approach has emerged as a powerful alternative for solving optimization and equilibrium problems. By modeling the solution process as an abstract Cauchy problem, this approach provides a deeper understanding of the asymptotic behavior of numerical algorithms and facilitates the design of new iterative schemes through time discretization [8, 9]. However, applying dynamical methods to EPs is particularly challenging due to the strict conditions required for the existence and uniqueness of global strong solutions [10, 11]. In a recent significant contribution, Anh, Hai and Dung [12] investigated a regularized dynamical system of the one-step type and proposed a corresponding relaxed iterative method for monotone equilibrium problems. That work demonstrated that combining regularization with a continuous-time framework can effectively ensure strong convergence of the generated trajectories to a specific solution. It is also to mention the projection techniques for equilibrium problems defined by a sum of two functions developed by Thuong and Vinh [13].

Building upon these recent advancements, this paper aims to further extend the scope of dynamical methods for EPs. We propose a new regularized dynamical system that incorporates an extragradient-type structure, distinguishing it from the one-step dynamics previously studied. Our approach focuses on establishing a rigorous mathematical foundation for the existence and uniqueness of global strong solutions under mild assumptions. Furthermore, we analyze the asymptotic properties of the system, proving the strong convergence of the trajectories toward the solution of the original problem. Finally, to bridge the gap between theory and practical implementation, we derive a two-step iterative algorithm via time discretization, integrated with a relaxation parameter to improve computational performance.

The remainder of this paper is structured as follows. Section 2 reviews essential definitions and preliminary results regarding regularized solution sequences. In Section 3, we introduce the proposed dynamical system and establish conditions for the existence, uniqueness, and strong convergence of its trajectories. The final section describes the development of a discrete iterative algorithm derived from the dynamical model, accompanied by a strong convergence analysis.

2. PRELIMINARIES

This section summarizes the fundamental definitions, notations, and technical results essential for the subsequent analysis. Throughout this study, H denotes a real Hilbert space equipped with the inner product $\langle\cdot,\cdot\rangle$ and the induced norm $\|\cdot\|$. The following standard properties hold for any $x, y \in H$ and parameters $\eta > 0$, $\tau \in (0,1)$:

$$\langle x, y\rangle \leq \|x\|\|y\| \leq \frac{1}{2}\left(\eta\|x\|^2 + \frac{1}{\eta}\|y\|^2\right); \tag{1}$$

$$\|\tau x + (1-\tau)y\|^2 \leq \tau\|x\|^2 + (1-\tau)\|y\|^2. \tag{2}$$

For the purpose of the following sections, we provide a summary of several key properties regarding the bifunction. $f: H \times H \to \mathbb{R}$ is said to be

(i) *monotone* on C if $f(x,y) + f(y,x) \leq 0$ for all $x, y \in C$.

(ii) γ-*strongly monotone* on C if $f(x,y) + f(y,x) \leq -\gamma\|x-y\|^2$ for all $x, y \in C$, where $\gamma > 0$ is a constant.

Next, we say that f satisfies a *L-Lipschitz-type condition* (LC) on C if there exists $L > 0$ such that $f(x,y) + f(y,z) \geq f(x,z) - L\|x-y\|\|y-z\| \quad \forall\, x, y, z \in C$. Another version, called *L-strongly Lipschitz-type condition* (SLC), is given by

$$|f(x,y) - f(z,y) + f(z,t) - f(x,t)| \leq L\|x-z\|\|y-t\| \;\forall\, x, y, z, t \in C.$$

It is evident that (SLC) implies (LC).

We now turn our attention to the concept of the proximal mapping. Let $F : C \to \mathbb{R}$ be a proper, convex, and lower semi-continuous function. Recalling the definition introduced in [7], the proximal operator $\text{prox}_F: H \to C$ associated with F is defined for each $u \in H$ as follows:

$$\text{prox}_F(x) = \text{argmin}\left\{F(t) + \frac{1}{2}\|x-t\|^2 : t \in C\right\}.$$

The subsequent analysis relies on a fundamental property of this mapping, as stated in the lemma below.

Lemma 2.1. ([7, Lemma 2.1]) *For any $x \in H$, $y \in C$, and parameter $\lambda > 0$, the proximal mapping satisfies the following inequality:*

$$\lambda\{F(y) - F(\text{prox}_{\lambda F}(x))\} \geq \langle x - \text{prox}_{\lambda F}(x), y - \text{prox}_{\lambda F}(x)\rangle. \tag{3}$$

Furthermore, the following technical result is essential for establishing the convergence of our iterative methods.

Lemma 2.2. ([14]) *Let $\{\Psi_n\}$ be a sequence of non-negative real numbers satisfying the recursive inequality $\Psi_{n+1} \leq (1-p_n)\Psi_n + q_n$ for all $n \geq 0$. Suppose that the sequences of real numbers $\{p_n\} \subset (0,1)$ and $\{q_n\}$ fulfill the conditions: $\lim_{n\to\infty} p_n = 0$, $\sum_{n=1}^{\infty} p_n = +\infty$ and $limsup_{n\to\infty} \frac{q_n}{p_n} \leq 0$. Then, it holds that $\lim_{n\to\infty} \Psi_n = 0$.*

3. REGULARIZED EXTRAGRADIENT-TYPE DYNAMICAL SYSTEM

Under the assumptions on H, C, and f detailed in Section 1, we present a novel regularized dynamical system of extragradient-type associated to EP(f, C) as follows:

$$\begin{cases} z(t) = \mathrm{prox}_{\lambda(t)(f(x(t),\cdot)+\alpha(t)g(x(t),\cdot))}\big(x(t)\big), \\ y(t) = \mathrm{prox}_{\lambda(t)(f(x(t),\cdot)+\alpha(t)g(z(t),\cdot))}\big(z(t)\big), \\ \dot{x}(t) + x(t) = y(t), \\ x(0) = x^0. \end{cases} \tag{4}$$

In (4), $\lambda(t) > 0$ and $\alpha(t) > 0$ for all $t \geq 0$ while x^0 is some fixed vector in C. $g: H \times H \to \mathbb{R}$ is an equilibrium bifunction on C selected to satisfy certain desired properties, which acts as a *regularizing bifunction*.

3.1. The existence and uniqueness of strong global solution

To begin with, we recall a definition introduced by [8]. A function $x(t): [0, +\infty) \to H$ is called a *strong global solution* to system (4) if it fulfills the following requirements:

(i) $x \in \mathrm{AC}_{\mathrm{loc}}([0, +\infty), H)$, which means that $x(t)$ is absolutely continuous on every compact interval $[0, T]$ for any $T > 0$;

(ii) System (4) is satisfied for almost every $t \in [0, +\infty)$;

(iii) The initial condition is satisfied, i.e., $x(0) = x^0$.

Let α^* and λ^* be two positive constants. Under the assumption presented below, we establish the existence and uniqueness of a strong global solution for the proposed dynamical system.

Assumption 3.1. We impose the following structural conditions throughout this analysis:

(A1) f satisfies an L-strongly Lipschitz-type condition on H;

(A2) The regularizing bifunction g is K-strongly Lipschitz-type on H;

(A3) For any fixed $x \in H$, both mappings $f(x,\cdot)$ and $g(x,\cdot)$ are convex, proper, lower semi-continuous, and sub-differentiable on C;

(A4) $g(x, y)$ is bounded whenever x and y belong to the bounded subsets of C;

(A5) The control mappings $\lambda(t): [0, +\infty) \to (0, \lambda^*)$ and $\alpha(t): [0, +\infty) \to (0, \alpha^*)$ are continuous time-dependent functions.

For the sake of conciseness, we denote $z_{\lambda,\alpha,x} \coloneqq \mathrm{prox}_{\lambda(t)(f(x(t),\cdot)+\alpha(t)g(x(t),\cdot))}(x)$, $T(\lambda, \alpha, x) \coloneqq \mathrm{prox}_{\lambda(t)(f(x(t),\cdot)+\alpha(t)g(z(t),\cdot))}\big(z_{\lambda,\alpha,x}\big)$ and $h(\lambda, \alpha, x) \coloneqq T(\lambda, \alpha, x) - x$. Utilizing these notations, the proposed dynamical system (4) can be equivalently rewritten as follows:

$$\begin{cases} \dot{x}(t) = h\big(\lambda(t), \alpha(t), x(t)\big), \\ x(0) = x^0. \end{cases} \tag{5}$$

To proceed with our analysis, several technical and auxiliary results are required.

Lemma 3.1. *Suppose that the equilibrium bifunctions* $f, g: H \times H \to \mathbb{R}$ *satisfy Assumption 3.1. For* $\lambda \in (0, \lambda^*]$ *and* $\alpha \in (0, \alpha^*]$*, the following properties are fulfilled.*

(i) *The mapping* $T(\lambda, \alpha, x)$ *is bounded for each* $x \in H$.
(ii) $T(\lambda, \alpha, x)$ *is continuous with respect to* α.
(iii) $T(\lambda, \alpha, x)$ *is continuous with respect to* λ.
(iv) *Let* $\alpha(t)$ *and* $\lambda(t)$ *be the continous time-dependent function defined in Assumption 3.1. Then, for any* $x \in H$*, the function* $\varphi_x(t) := \|h(\lambda(t), \alpha(t), x)\|$ *is continuous on the interval* $[0, +\infty)$.

Proof. (i) For any fixed $x \in H$, we write $z_{\lambda,\alpha}$ instead of $z_{\lambda,\alpha,x}$ for brevity. By invoking Lemma 2.1, for any $u \in C$, it follows that

$$\begin{aligned}&\langle u - T(\lambda, \alpha, x), z_{\lambda,\alpha} - T(\lambda, \alpha, x)\rangle \\ &\leq \lambda\big[f(x, u) + \alpha g\big(z_{\lambda,\alpha}, u\big)\big] - \lambda\Big[f\big(x, T(\lambda, \alpha, x)\big) + \alpha g\Big(z_{\lambda,\alpha}, T(\lambda, \alpha, x)\Big)\Big],\end{aligned}$$

which can be equivalently written as

$$\begin{aligned}\|T(\lambda, \alpha, x)\|^2 \leq \lambda\Big[&f(x, u) + \alpha g\big(z_{\lambda,\alpha}, u\big) - f\big(x, T(\lambda, \alpha, x)\big) - \alpha g\Big(z_{\lambda,\alpha}, T(\lambda, \alpha, x)\Big)\Big] \\ &+\langle u + z_{\lambda,\alpha}, T(\lambda, \alpha, x)\rangle - \langle u, z_{\lambda,\alpha}\rangle \qquad (6)\end{aligned}$$

Alternatively, from the L-strongly Lipschitz-type property of f, it is easy to see that

$$-f\big(x, T(\lambda, \alpha, x)\big) \leq -f\big(x, z_{\lambda,\alpha}\big) - f\Big(z_{\lambda,\alpha}, T(\lambda, \alpha, x)\Big) + L\big\|z_{\lambda,\alpha} - x\big\|\big\|z_{\lambda,\alpha} - T(\lambda, \alpha, x)\big\|.$$

A similar argument yields

$$f(x, u) - f\big(x, z_{\lambda,\alpha}\big) \leq -f\big(u, z_{\lambda,\alpha}\big) + L\|x - u\|\big\|u - z_{\lambda,\alpha}\big\|.$$

Combining the previous two inequalities with (6), we deduce that

$$\begin{aligned}\|T(\lambda, \alpha, x)\|^2 &\leq \lambda\big[f(x, u) + \alpha g\big(z_{\lambda,\alpha}, u\big) - f\big(x, z_{\lambda,\alpha}\big)\big] \\ &\quad -\lambda\Big[f\Big(z_{\lambda,\alpha}, T(\lambda, \alpha, x)\Big) + \alpha g\Big(z_{\lambda,\alpha}, T(\lambda, \alpha, x)\Big)\Big] \\ &\quad +\lambda L\big\|z_{\lambda,\alpha} - x\big\|\big\|z_{\lambda,\alpha} - T(\lambda, \alpha, x)\big\| + \langle u + z_{\lambda,\alpha}, T(\lambda, \alpha, x)\rangle - \langle u, z_{\lambda,\alpha}\rangle \\ &\leq \lambda\big[\alpha g\big(z_{\lambda,\alpha}, u\big) - f\big(u, z_{\lambda,\alpha}\big)\big] \\ &\quad -\lambda\Big[f\Big(z_{\lambda,\alpha}, T(\lambda, \alpha, x)\Big) + \alpha g\Big(z_{\lambda,\alpha}, T(\lambda, \alpha, x)\Big)\Big] + \lambda L\|x - u\|\big\|u - z_{\lambda,\alpha}\big\| \\ &\quad +\lambda L\big\|z_{\lambda,\alpha} - x\big\|\big\|z_{\lambda,\alpha} - T(\lambda, \alpha, x)\big\| + \langle u + z_{\lambda,\alpha}, T(\lambda, \alpha, x)\rangle - \langle u, z_{\lambda,\alpha}\rangle \\ &\leq \big(\lambda L\big\|z_{\lambda,\alpha} - x\big\| + \big\|u + z_{\lambda,\alpha}\big\|\big)\|T(\lambda, \alpha, x)\| \\ &\quad -\lambda\Big[f\Big(z_{\lambda,\alpha}, T(\lambda, \alpha, x)\Big) + \alpha g\Big(z_{\lambda,\alpha}, T(\lambda, \alpha, x)\Big)\Big] + \xi, \qquad (7)\end{aligned}$$

where $\xi = \left|\lambda\alpha g\left(z_{\lambda,\alpha}, u\right) - \lambda f\left(u, z_{\lambda,\alpha}\right) - \langle u, z_{\lambda,\alpha}\rangle\right|$. Owing to the convexity of the mappings $f(x,\cdot)$ and $g(x,\cdot)$ on C, one can find the vectors $A = A\left(z_{\lambda,\alpha}\right) \in H$ and $B = B\left(z_{\lambda,\alpha}\right) \in H$, along with scalars $E = E\left(z_{\lambda,\alpha}\right)$ and $F = F\left(z_{\lambda,\alpha}\right)$ satisfying the following inequalites

$$f\left(z_{\lambda,\alpha}, w\right) \geq \langle A, w\rangle + E, \quad g\left(z_{\lambda,\alpha}, w\right) \geq \langle B, w\rangle + F \quad \forall w \in C. \tag{8}$$

A combination of (8) and (7) leads to the following relation:

$$\begin{aligned}\|T(\lambda,\alpha,x)\|^2 &\leq \left(\lambda L\left\|z_{\lambda,\alpha} - x\right\| + \left\|u + z_{\lambda,\alpha}\right\|\right)\|T(\lambda,\alpha,x)\| \\ &\quad -\lambda(\langle A, T(\lambda,\alpha,x)\rangle + \alpha\langle B, T(\lambda,\alpha,x)\rangle) - \lambda(E + \alpha F) + \xi \\ &\leq \left(\lambda L\left\|z_{\lambda,\alpha} - x\right\| + \left\|u + z_{\lambda,\alpha}\right\|\right)\|T(\lambda,\alpha,x)\| \\ &\quad +\lambda(\|A\| + \|B\|)\|T(\lambda,\alpha,x)\| + \lambda|E + \alpha F| + \xi. \end{aligned} \tag{9}$$

Following the arguments in the proof of [12, Lemma 2.2], $z_{\lambda,\alpha}$ is bounded and continuous with respect to both $\lambda \in (0, \lambda^*]$ and $\alpha \in (0, \alpha^*]$. Thus, inequality (9) guarantees the boundedness of $T(\lambda,\alpha,x)$.

(ii) For fixed λ, α and x, Lemma 2.1 implies that for any $\beta \in (0, \alpha^*]$, we have

$$\begin{aligned}&\langle T(\lambda,\alpha,x) - T(\lambda,\beta,x), T(\lambda,\alpha,x) - z_{\lambda,\alpha}\rangle \\ &\leq \lambda\left[f\left(x, T(\lambda,\beta,x)\right) + \alpha g\left(z_{\lambda,\alpha}, T(\lambda,\beta,x)\right)\right] - \lambda\left[f\left(x, T(\lambda,\alpha,x)\right) + \alpha g\left(z_{\lambda,\alpha}, T(\lambda,\alpha,x)\right)\right]\end{aligned}$$

and

$$\begin{aligned}&\langle T(\lambda,\beta,x) - T(\lambda,\alpha,x), T(\lambda,\beta,x) - z_{\lambda,\beta}\rangle \\ &\leq \lambda\left[f\left(x, T(\lambda,\alpha,x)\right) + \beta g\left(z_{\lambda,\beta}, T(\lambda,\alpha,x)\right)\right] - \lambda\left[f\left(x, T(\lambda,\beta,x)\right) + \beta g\left(z_{\lambda,\beta}, T(\lambda,\beta,x)\right)\right].\end{aligned}$$

Summing these two inequalities, we obtain

$$\begin{aligned}&\|T(\lambda,\alpha,x) - T(\lambda,\beta,x)\|^2 + \langle T(\lambda,\beta,x) - T(\lambda,\alpha,x), z_{\lambda,\alpha} - z_{\lambda,\beta}\rangle \\ &\leq \lambda\beta\left[g\left(z_{\lambda,\alpha}, T(\lambda,\beta,x)\right) - g\left(z_{\lambda,\alpha}, T(\lambda,\alpha,x)\right) + g\left(z_{\lambda,\beta}, T(\lambda,\alpha,x)\right) - g\left(z_{\lambda,\beta}, T(\lambda,\beta,x)\right)\right] \\ &\quad +\lambda(\alpha - \beta)\left[g\left(z_{\lambda,\alpha}, T(\lambda,\beta,x)\right) - g\left(z_{\lambda,\alpha}, T(\lambda,\alpha,x)\right)\right].\end{aligned}$$

From the last relation, using the (SLC) of g, we imply that

$$\begin{aligned}\|T(\lambda,\alpha,x) - T(\lambda,\beta,x)\|^2 &\leq (K+1)\left\|z_{\lambda,\alpha} - z_{\lambda,\beta}\right\|\|T(\lambda,\beta,x) - T(\lambda,\alpha,x)\| \\ &\quad +\lambda(\alpha - \beta)\left[g\left(z_{\lambda,\alpha}, T(\lambda,\beta,x)\right) - g\left(z_{\lambda,\alpha}, T(\lambda,\alpha,x)\right)\right].\end{aligned} \tag{10}$$

Since $T(\lambda,\alpha,x)$ and $T(\lambda,\beta,x)$ are bounded, the values $g\left(z_{\lambda,\alpha}, T(\lambda,\beta,x)\right)$ and $g\left(z_{\lambda,\alpha}, T(\lambda,\alpha,x)\right)$ are also bounded. Letting $\beta \to \alpha$, it follows from [12, Lemma 2.2] that $\left\|z_{\lambda,\alpha} - z_{\lambda,\beta}\right\| \to 0$. Consequently, taking the limit in (10) yields $\lim_{\beta\to\alpha} T(\lambda,\beta,x) = T(\lambda,\alpha,x)$, which confirms the continuity of $T(\lambda,\alpha,x)$ with respect to α.

(iii) Fix λ, α and x. For all $\gamma \in (0, \lambda^*]$, we imply from Lemma 2.1 that,

$$\begin{aligned}&\langle T(\lambda,\alpha,x) - T(\gamma,\alpha,x), T(\lambda,\alpha,x) - z_{\lambda,\alpha}\rangle \\ &\leq \lambda\left[f\big(x, T(\gamma,\alpha,x)\big) + \alpha g\left(z_{\lambda,\alpha}, T(\gamma,\alpha,x)\right)\right] - \lambda\left[f\big(x, T(\lambda,\alpha,x)\big) + \alpha g\left(z_{\lambda,\alpha}, T(\lambda,\alpha,x)\right)\right]\end{aligned}$$

and

$$\begin{aligned}&\langle T(\gamma,\alpha,x) - T(\lambda,\alpha,x), T(\gamma,\alpha,x) - z_{\gamma,\alpha}\rangle \\ &\leq \gamma\left[f\big(x, T(\lambda,\alpha,x)\big) + \alpha g\left(z_{\gamma,\alpha}, T(\lambda,\alpha,x)\right)\right] - \gamma\left[f\big(x, T(\lambda,\alpha,x)\big) + \alpha g\left(z_{\gamma,\alpha}, T(\lambda,\beta,x)\right)\right].\end{aligned}$$

An addition of these inequalities yields

$$\begin{aligned}&\|T(\lambda,\alpha,x) - T(\gamma,\alpha,x)\|^2 + \langle T(\gamma,\alpha,x) - T(\lambda,\alpha,x), z_{\lambda,\alpha} - z_{\gamma,\alpha}\rangle \\ &\leq \gamma\alpha\left[g\left(z_{\gamma,\alpha}, T(\lambda,\alpha,x)\right) - g\left(z_{\gamma,\alpha}, T(\gamma,\alpha,x)\right) + g\left(z_{\lambda,\alpha}, T(\gamma,\alpha,x)\right) - g\left(z_{\lambda,\alpha}, T(\lambda,\alpha,x)\right)\right] \\ &\quad +(\lambda-\gamma)\alpha\left[g\left(z_{\lambda,\alpha}, T(\gamma,\alpha,x)\right) - g\left(z_{\lambda,\alpha}, T(\lambda,\alpha,x)\right)\right].\end{aligned}$$

As a result, by combining the aforementioned estimates with the boundedness of the mappings involved, a similar limiting process as in (ii) establishes that $\lim_{\gamma\to\lambda} T(\gamma,\alpha,x) = T(\lambda,\alpha,x)$. This directly confirms the continuity of $T(\lambda,\alpha,x)$ with respect to λ.

(iv) Given $x \in H$ and $t_0 \geq 0$. For all $t \geq 0$, we see that

$$\begin{aligned}|\varphi_x(t) - \varphi_x(t_0)| &\leq \|h(\lambda(t),\alpha(t),x) - h(\lambda(t_0),\alpha(t_0),x)\| \\ &= \|T(\lambda(t),\alpha(t),x) - T(\lambda(t_0),\alpha(t_0),x)\| \\ &\leq \|T(\lambda(t),\alpha(t),x) - T(\lambda(t),\alpha(t_0),x)\| \\ &\quad +\|T(\lambda(t),\alpha(t_0),x) - T(\lambda(t_0),\alpha(t_0),x)\|. \qquad (11)\end{aligned}$$

By the virtue of relation (10), it follows that

$$\begin{aligned}&\|T(\lambda(t),\alpha(t),x) - T(\lambda(t),\alpha(t_0),x)\|^2 \\ &\leq (K+1)\left\|z_{\lambda(t),\alpha(t)} - z_{\lambda(t),\alpha(t_0)}\right\|\|T(\lambda(t),\alpha(t),x) - T(\lambda(t),\alpha(t_0),x)\| \\ &\quad +\lambda(t)\big(\alpha(t) - \alpha(t_0)\big)\left[g\left(z_{\lambda(t),\alpha(t_0)}, T(\lambda(t),\alpha(t),x)\right) - g\left(z_{\lambda(t),\alpha(t)}, T(\lambda(t),\alpha(t),x)\right)\right].\end{aligned}$$

Taking account to the continuity of $\alpha(t)$ and $z_{\lambda(t),\alpha(t)}$, along with the boundedness of g guaranteed by Assumption 3.1. it follows from the last inequality that

$$\|T(\lambda(t),\alpha(t),x) - T(\lambda(t),\alpha(t_0),x)\| \to 0 \text{ as } t \to t_0. \qquad (12)$$

Furthermore, based on the continuity established in part (iii), it holds that

$$\|T(\lambda(t),\alpha(t_0),x) - T(\lambda(t_0),\alpha(t_0),x)\| \to 0 \text{ as } t \to t_0. \qquad (13)$$

Combining (11), (12) and (13) immediately yields $|\varphi_x(t) - \varphi_x(t_0)| \to 0$ as $t \to t_0$, which completes the proof of the continuity of $\varphi_x(t)$ on $[0, +\infty)$. ■

Lemma 3.2. *Under Assumption 3.1. for any fixed λ, α, the mapping $U: H \to C$ defined by*

$$U(x) = \underset{u\in C}{argmin}\left\{\lambda f(x,u) + \lambda\alpha g\left(z_{\lambda,\alpha,x}, u\right) + \frac{1}{2}\left\|u - z_{\lambda,\alpha,x}\right\|^2\right\}$$

is Lipschitz continuous on H.

Proof. For any fixed parameters λ and α, we write z_x instead of $z_{\lambda,\alpha,x}$ for conciseness. Recall from [12, Lemma 2.3] that the mapping $x \to z_x$ is $(L+1)$-Lipschitz continuous on H. Let $x, y \in H$. By applying Lemma 2.1, it follows that

$$\langle z_x - U(x), U(y) - U(x)\rangle \le \lambda\left[f\left(x, U(y)\right) + \alpha g\left(z_x, U(y)\right) - f\left(x, U(x)\right) - g\left(z_x, U(x)\right)\right]$$

and

$$\langle z_y - U(y), U(x) - U(y)\rangle \le \lambda\left[f\left(y, U(x)\right) + \alpha g\left(z_y, U(x)\right) - f\left(y, U(y)\right) - g\left(z_y, U(y)\right)\right].$$

Summing these two inequalities yields

$$\begin{aligned}&\|U(y) - U(x)\|^2 + \langle U(y) - U(x), z_x - z_y\rangle\\ &\le \lambda\left[f\left(x, U(y)\right) - f\left(x, U(x)\right) + f\left(y, U(x)\right) - f\left(y, U(y)\right)\right]\\ &\quad +\lambda\alpha\left[g\left(z_x, U(y)\right) - g\left(z_x, U(x)\right) + g\left(z_y, U(x)\right) - g\left(z_y, U(y)\right)\right].\end{aligned}$$

Rearranging the terms in the last relation and exploiting the (SLCs) of both f and g, alongside the Lipschitz continuity of z_x, we arrive at

$$\begin{aligned}\|U(y) - U(x)\|^2 &\le \lambda L\|x - y\|\|U(x) - U(y)\| + (\lambda\alpha K + 1)\left\|z_x - z_y\right\|\|U(x) - U(y)\|\\ &\le [\lambda L + (\lambda\alpha K + 1)(L+1)]\|U(x) - U(y)\|\|x - y\|,\end{aligned}$$

which subsequently implies that $\|U(y) - U(x)\| \le [\lambda L + (\lambda\alpha K + 1)(L+1)]\|x - y\|$. Consequently, we conclude that the mapping U is $\overline{M}$-Lipschitz continuous on H, with the Lipschitz constant $\overline{M} = [\lambda L + (\lambda\alpha K + 1)(L+1)]$.

Next, we formulate and demonstrate the existence and uniqueness of strong global solutions.

Theorem 3.1. *Supposing that Assumption 3.1 is satisfied and that a unique solution exists for the problem* $Sol\left(g, Sol(f, C)\right)$*, it follows that for every* $x_0 \in C$*, there exists a unique strong global solution to the dynamical system* (5).

Proof. By employing the Cauchy-Lipschitz-Picard theorem (see, e.g., [15, Proposition 6.2.1]), the proof can be carried out in an identical manner to that of [12, Theorem 3.1]. The key ingredients are the continuity of φ_x and the Lipschitz continuity of $U(x)$, both of which were previously verified in Lemma 3.1 and Lemma 3.2.

3.2. Trajectory convergence analysis

We first introduce the underlying structural requirements for our convergence analysis.

Assumption 3.2. The following conditions are assumed throughout this subsection:

(B1) The equilibrium bifunction f is monotone and fulfills an L-Lipschitz-type condition on the set C.

(B2) The regularizing bifunction g is γ-strongly monotone and satisfies a K-Lipschitz-type condition on C.

(B3) For each pair $x, y \in C$, the mapping $g(x,\cdot)$ is proper, convex, lower semi-continuous, and sub-differentiable on C. Both $\partial g(x,\cdot)(x)$ and $g(\cdot, y)$ are bounded on any bounded subsets of C. Furthermore, $f(x,\cdot)$ is proper, convex, lower semi-continuous, and sub-differentiable, while $f(\cdot, y)$ and $g(\cdot, y)$ are weakly upper semi-continuous on C.

(B4) The stepsize parameter $\lambda(t)$: $[0,+\infty) \to (0,+\infty]$ is a continuous function, and the regularization parameter $\alpha(t)$: $[0,+\infty) \to (0,+\infty]$ is continuously differentiable, such that the following asymptotic conditions are satisfied:

$$\lim_{t\to+\infty} \alpha(t) = 0\,;\ \int_0^{+\infty} \lambda(t)\alpha(t)dt = +\infty;\ \lim_{t\to+\infty} \frac{\lambda(t)}{\alpha(t)} = 0 \ \text{ and } \lim_{t\to+\infty} \frac{\dot{\alpha}(t)}{\lambda(t)\alpha^2(t)} = 0.$$

Remark 3.1. A concrete instance of functions $\alpha(t)$ and $\lambda(t)$ that comply with condition (B4) under Assumption 3.2 is given by $\lambda(t) = \frac{1}{(t+1)^p}$ and $\alpha(t) = \frac{1}{(t+1)^q}$, with some fixed numbers p and q such that $0 < q < p < 1 - q$. For example, by choosing $p = \frac{1}{2}$ and $q = \frac{1}{4}$, all the required conditions are satisfied.

Theorem 3.2. *Provided that Assumption 3.2 holds and system* (5) *possesses a unique strong global solution* $x(t)$. *Then* $\lim_{t\to+\infty} x(t) = \hat{x}$. *Here,* $\hat{x}$ *is the unique solution of the bilevel equilibrium problem* $EP\big(g, Sol(f, C)\big)$.

Proof. By Assumption 3.2 (B2), the bifunction $f + \alpha(t)g$ is strongly monotone for each $t > 0$. This ensures that the problem EP$(f + \alpha(t)g, C)$ possesses a unique solution, denoted by $x_{\alpha(t)}$. According to [12, Lemma 2.4], $x_{\alpha(t)}$ is differentiable with respect to t almost everywhere on $[0,+\infty)$ and there exists $M > 0$ such that

$$\left\|\frac{d}{dt} x_{\alpha(t)}\right\| \le M \frac{|\dot{\alpha}(t)|}{\alpha(t)} \ \ \forall t \ge 0. \tag{14}$$

Furthermore, also by [12, Lemma 2.4], it holds that $\lim_{t\to+\infty} x_{\alpha(t)} = \hat{x}$, where $\hat{x}$ is the unique solution to EP$\big(g, Sol(f, C)\big)$.

Next, we observe the following algebraic identity:

$$\begin{aligned}\langle x(t) - x_{\alpha(t)}, y(t) - x(t)\rangle = {} & \langle y(t) - x_{\alpha(t)}, y(t) - z(t)\rangle + \langle z(t) - x_{\alpha(t)}, z(t) - x(t)\rangle \\ & + \langle y(t) - z(t), z(t) - x(t)\rangle - \|x(t) - y(t)\|^2.\end{aligned} \tag{15}$$

In addition, it holds that

$$\begin{aligned}\langle y(t) - x_{\alpha(t)}, y(t) - z(t)\rangle \le {} & \lambda(t)\big[f\big(x(t), x_{\alpha(t)}\big) - f\big(x(t), y(t)\big)\big] \\ & + \lambda(t)\alpha(t)\big[g\big(z(t), x_{\alpha(t)}\big) - g\big(z(t), y(t)\big)\big].\end{aligned}$$

By combining this relation with the Lipschitz-type conditions of both of f, g, alongside the fact that $x_{\alpha(t)}$ solves the regularized problem EP$(f+\alpha(t)g, C)$, we find that

$$\begin{aligned}\langle y(t)-x_{\alpha(t)}, y(t)-z(t)\rangle \le\ & \lambda(t)L\|x(t)-y(t)\|\left\|x_{\alpha(t)}-y(t)\right\| \\ & +\lambda(t)\alpha(t)K\|z(t)-y(t)\|\left\|x_{\alpha(t)}-y(t)\right\| \\ & -\gamma\lambda(t)\alpha(t)K\left\|x_{\alpha(t)}-y(t)\right\|^2. \end{aligned} \quad (16)$$

Using the same argument, we obtain

$$\begin{aligned}\langle z(t)-x_{\alpha(t)}, z(t)-x(t)\rangle \le\ & \lambda(t)(L+\alpha(t)K)\|x(t)-z(t)\|\left\|x_{\alpha(t)}-z(t)\right\| \\ & -\gamma\lambda(t)\alpha(t)K\left\|x_{\alpha(t)}-y(t)\right\|^2. \end{aligned} \quad (17)$$

It is also easy to see that

$$\begin{aligned}\langle y(t)-z(t), z(t)-x(t)\rangle &= \langle y(t)-z(t), y(t)-x(t)\rangle - \|y(t)-z(t)\|^2 \\ &\le \frac{1}{2}\|x(t)-y(t)\|^2 - \frac{1}{2}\|y(t)-z(t)\|^2. \end{aligned} \quad (18)$$

In addition, using (1) and (2), the following inequalities hold:

$$\lambda(t)L\|x(t)-y(t)\|\left\|x_{\alpha(t)}-y(t)\right\| \le \frac{\|x(t)-y(t)\|^2}{8} + 2\lambda(t)^2L^2\left\|x_{\alpha(t)}-y(t)\right\|^2, \quad (19)$$

$$\begin{aligned}\lambda(t)\alpha(t)K\|x(t)-y(t)\|\left\|x_{\alpha(t)}-y(t)\right\| \le\ & \frac{\|z(t)-y(t)\|^2}{4} \\ & +\lambda(t)^2\alpha(t)^2K^2\left\|x_{\alpha(t)}-y(t)\right\|^2, \end{aligned} \quad (20)$$

$$\begin{aligned}\lambda(t)(L+\alpha(t)K)\|x(t)-z(t)\|\left\|x_{\alpha(t)}-z(t)\right\| \le\ & \frac{\|x(t)-y(t)\|^2}{4} + \frac{\|y(t)-z(t)\|^2}{4} \\ & +2\lambda(t)^2(L+\alpha(t)K)^2\left\|x_{\alpha(t)}-z(t)\right\|^2, \end{aligned} \quad (21)$$

and

$$-\gamma\lambda(t)\alpha(t)\left\|x_{\alpha(t)}-y(t)\right\|^2 \le -\gamma\lambda(t)\alpha(t)\left(\frac{1}{2}\left\|x(t)-x_{\alpha(t)}\right\|^2 - \|x(t)-y(t)\|^2\right). \quad (22)$$

Synthesizing relations(15)-(22) leads to

$$\begin{aligned}\langle x(t)-x_{\alpha(t)}, y(t)-x(t)\rangle \le\ & \left(2\lambda(t)^2L^2+\lambda(t)^2\alpha(t)^2K^2-\gamma\lambda(t)\alpha(t)\right)\left\|x_{\alpha(t)}-y(t)\right\|^2 \\ & -\frac{1}{8}\|x(t)-y(t)\|^2 \\ & +[2\lambda(t)^2(L+\alpha(t)K)^2-\gamma\lambda(t)\alpha(t)]\left\|x_{\alpha(t)}-y(t)\right\|^2. \end{aligned} \quad (23)$$

For t sufficiently large such that $2\lambda(t)^2L^2+\lambda(t)^2\alpha(t)^2K^2-\gamma\lambda(t)\alpha(t) \le -\frac{1}{2}\gamma\lambda(t)\alpha(t)$ and $\gamma\lambda(t)\alpha(t) \le 0.25$, applying these bounds to (23) demonstrates that

$$\begin{aligned}
&\langle x(t) - x_{\alpha(t)}, y(t) - x(t)\rangle \\
&\leq -\frac{1}{2}\gamma\lambda(t)\alpha(t)\left\|x_{\alpha(t)} - y(t)\right\|^2 - \frac{1}{8}\|x(t) - y(t)\|^2 \\
&\leq -\frac{1}{2}\gamma\lambda(t)\alpha(t)\left(\frac{1}{2}\left\|x(t) - x_{\alpha(t)}\right\|^2 - \|x(t) - y(t)\|^2\right) - \frac{1}{8}\|x(t) - y(t)\|^2 \\
&= -\frac{1}{4}\gamma\lambda(t)\alpha(t)\left\|x(t) - x_{\alpha(t)}\right\|^2 + \left(\frac{1}{2}\gamma\lambda(t)\alpha(t) - \frac{1}{8}\right)\|x(t) - y(t)\|^2 \\
&\leq -\frac{1}{4}\gamma\lambda(t)\alpha(t)\left\|x(t) - x_{\alpha(t)}\right\|^2. \qquad (24)
\end{aligned}$$

We now let $V(t) \coloneqq \frac{1}{2}\left\|x(t) - x_{\alpha(t)}\right\|^2$. From (4), we see that $\dot{x}(t) = y(t) - x(t)$. Thus, we have

$$\begin{aligned}
\dot{V}(t) &= \langle x(t) - x_{\alpha(t)}, \dot{x}(t) - \dot{x}_{\alpha(t)}\rangle = \langle x(t) - x_{\alpha(t)}, y(t) - x(t) - \dot{x}_{\alpha(t)}\rangle \\
&= \langle x(t) - x_{\alpha(t)}, y(t) - x(t)\rangle - \langle x(t) - x_{\alpha(t)}, y(t) - x(t)\rangle. \qquad (25)
\end{aligned}$$

Applying inequalities (1), (2), (14) and (24) to (25) yields

$$\dot{V}(t) \leq -\frac{1}{4}\gamma\lambda(t)\alpha(t)\left\|x(t) - x_{\alpha(t)}\right\|^2 + \left(2M^2\frac{|\dot{\alpha}(t)|^2}{\gamma\lambda(t)\alpha(t)^3} + \frac{1}{8}\gamma\lambda(t)\alpha(t)\left\|x(t) - x_{\alpha(t)}\right\|^2\right),$$

or equivalently,

$$\dot{V}(t) + \frac{1}{4}\gamma\lambda(t)\alpha(t)V(t) \leq \frac{2M^2}{\gamma}\cdot\frac{|\dot{\alpha}(t)|^2}{\lambda(t)\alpha(t)^3}. \qquad (26)$$

By integrating (26), we obtain the following explicit bound for the functional $V(t)$:

$$V(t) \leq e^{-\frac{1}{4}\gamma\int_0^t \lambda(r)\alpha(r)dr}\left(\frac{2M^2}{\gamma}\int_0^t e^{\frac{1}{4}\gamma\int_0^s \lambda(r)\alpha(r)dr}\frac{|\dot{\alpha}(s)|^2}{\lambda(s)\alpha(s)^3}ds + V(0)\right).$$

Taking into account the condition $\int_0^{+\infty}\lambda(r)\alpha(r)dr = +\infty$, the asymptotic behavior $V(t) \to 0$ as $t \to +\infty$ is guaranteed if we are able to point out that

$$\lim_{t\to+\infty}\frac{\int_0^t e^{\frac{1}{4}\gamma\int_0^s \lambda(r)\alpha(r)dr}\frac{|\dot{\alpha}(s)|^2}{\lambda(s)\alpha(s)^3}ds}{e^{\frac{1}{4}\gamma\int_0^t \lambda(r)\alpha(r)dr}} = 0.$$

The verification is straightforward if $\int_0^{+\infty} e^{\frac{1}{4}\gamma\int_0^s \lambda(r)\alpha(r)dr}\frac{|\dot{\alpha}(t)|^2}{\lambda(t)\alpha(t)^3}ds < +\infty$. Alternatively, if the integral diverges, an application of L'Hôpital's rule yields

$$\lim_{t\to+\infty}\frac{\int_0^t e^{\frac{1}{4}\gamma\int_0^s\lambda(r)\alpha(r)dr}\dfrac{|\dot{\alpha}(s)|^2}{\lambda(s)\alpha(s)^3}ds}{e^{\frac{1}{4}\gamma\int_0^t\lambda(r)\alpha(r)dr}}=\lim_{t\to+\infty}\frac{e^{\frac{1}{4}\gamma\int_0^t\lambda(r)\alpha(r)dr}\dfrac{|\dot{\alpha}(t)|^2}{\lambda(t)\alpha(t)^3}}{e^{\frac{1}{4}\gamma\int_0^t\lambda(r)\alpha(r)dr}\dfrac{1}{4}\gamma\lambda(t)\alpha(t)}.$$

$$=\lim_{t\to+\infty}\frac{4|\dot{\alpha}(t)|^2}{\gamma\lambda(t)^2\alpha(t)^4}=0.$$

Hence, it follows that $V(t)\to 0$ as $t\to+\infty$, or equivalently, $\lim_{t\to+\infty}x(t)=\lim_{t\to+\infty}x_{\alpha(t)}=\hat{x}$, which concludes the proof. ■

3.3. Discretized iterative methods

By applying a forward Euler discretization scheme to the continuous dynamical system (5), where the time derivative is approximated by $\dot{x}(t_k)\approx\frac{x^{k+1}-x^k}{\tau_k}$, we can derive the following regularized extragradient-type iterative method integrated with relaxation parameters, namely:

REA1 (Regularized Extragradient-type Algorithm 1)

$$\begin{cases}z^k &= \mathrm{prox}_{\lambda_k\left(f(x^k,\cdot)+\alpha_k g(x^k,\cdot)\right)}(x_k),\\ y^k &= \mathrm{prox}_{\lambda_k\left(f(x^k,\cdot)+\alpha_k g(z^k,\cdot)\right)}(z_k),\\ x^{k+1} &= (1-\tau_k)x^k+\tau_k y^k.\end{cases}$$

Furthermore, inspired by the multi-stage parameter-shifted mechanisms widely used in discrete splitting algorithms, we also introduce an alternative numerical variant:

REA2 (Regularized Extragradient-type Algorithm 2)

$$\begin{cases}z^k &= \mathrm{prox}_{\lambda_k\left(f(x^k,\cdot)+\alpha_k g(x^k,\cdot)\right)}(x_k),\\ y^k &= \mathrm{prox}_{\lambda_{k+1}\left(f(x^k,\cdot)+\alpha_{k+1} g(z^k,\cdot)\right)}(z_k),\\ x^{k+1} &= (1-\tau_k)x^k+\tau_k y^k.\end{cases}$$

Theorem 3.3. *Suppose that all conditions specified in Theorem 3.2 hold for the bifunctions f and g. Furthermore, we also assume that $\{\tau_k\}\subset[a,b]$ for a pair of fixed scalars a, b satisfying $0<a<b<1$ and the sequences $\{\lambda_k\}$, $\{\alpha_k\}\subset(0,+\infty)$ fulfill the following conditions:*

$$\sum_{k=0}^{+\infty}\lambda_k\alpha_k=+\infty;\ \lim_{k\to\infty}\alpha_k=0;\ \lim_{k\to\infty}\frac{\lambda_k}{\alpha_k}=0;\ \lim_{k\to\infty}\frac{|\alpha_{k+1}-\alpha_k|}{\lambda_k\alpha_k^2}=0\,.$$

It then follows that $\lim_{k\to\infty}x_k=\hat{x}$, where x denotes the unique solution to the bilevel problem $EP\big(g,Sol(f,C)\big)$.

Remark 3.2. An instance of sequences $\{\lambda_k\}$ and $\{\alpha_k\}$ satisfying the required conditions in Theorem 3.3 is given by $\lambda_k=(k+1)^{-p}$ and $\lambda_k=(k+1)^{-q}$ with $0<q<p<1-q$. For example, we can take $p=\frac{1}{2}$ and $q=\frac{1}{4}$. From an algorithmic viewpoint, the condition that

$\lim_{k\to\infty}\frac{\lambda_k}{\alpha_k}=0$ balances the extragradient stepsize and the regularization parameter, ensuring that $O(\lambda_k^2)$ are asymptotically dominated by $O(\lambda_k\alpha_k)$ to guarantee strong convergence.

Sketch of the proof. We can prove that $\lim_{k\to\infty}\|x^k - x_{\alpha_k}\| = 0$ and thus

$$\lim_{k\to\infty} x^k = \lim_{k\to\infty} x_{\alpha_k} = \hat{x} \in Sol\big(g, Sol(f, C)\big).$$

From this, the strong convergence of the REA1 and REA2 algorithms is immediately deduced. For the sake of conciseness, the detailed algebraic proofs are not presented here. The complete derivation is available online for readers and reviewers at [16].

4. NUMERICAL EXPERIMENTS

In this section, we validate the computational efficiency of the proposed algorithms via MATLAB Online on a 64-bit cloud computing platform.

Example 4.1. Consider the bilevel equilibrium problem described in Section 3, where the space is $\mathbb{R}^5$. The feasible constraint set is defined as:

$$C = \left\{x \in \mathbb{R}^5 : \sum_{i=1}^{5} x_i \geq 1, \text{ and } -5 \leq x_i \leq 5 \ \forall\, i = 1, 2, \dots, 5\right\}.$$

The equilibrium bifunction $f(x,y)$ is given by $f(x,y) = \langle Px + Qy + q, y - x\rangle$, where the vector $q = (1,-2,-1,2,-1)^T$ and the matrices P, Q are given as follows

$$P = \begin{bmatrix} 3.1 & 2 & 0 & 0 & 0 \\ 2 & 3.6 & 0 & 0 & 0 \\ 0 & 0 & 3.5 & 2 & 0 \\ 0 & 0 & 2 & 3.5 & 0 \\ 0 & 0 & 0 & 0 & 3 \end{bmatrix}, \quad Q = \begin{bmatrix} 1.6 & 1 & 0 & 0 & 0 \\ 1 & 1.6 & 0 & 0 & 0 \\ 0 & 0 & 1.5 & 1 & 0 \\ 0 & 0 & 1 & 1.5 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}.$$

Additionally, the regularizing is chosen as $g(x,y) = \langle x, y - x\rangle$. To satisfy the theoretical convergence criteria established in Theorem 3.3, the dynamic stepsize and regularization parameter sequences are set to $\lambda_k = k^{-0.5}$ and $\alpha_k = k^{-0.25}$, respectively. We also let $\tau_k = 0.01$ for every $k \geq 1$.

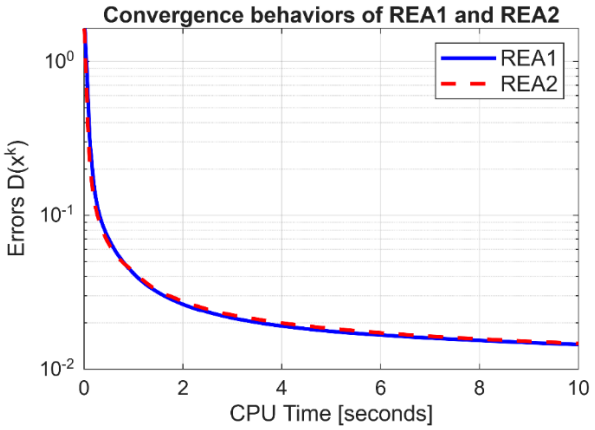


Figure 4.1. Algorithms with $\lambda_k = k^{-0.5}$ and $\alpha_k = k^{-0.25}$. The number of iterations is 13282 and 12521, respectively

The numerical convergence process is adaptively tracked via the computational residual error function $D(x^k) = \left\|x^k - \text{prox}_{0.1f(x^k,\cdot)}(x^k)\right\| = \left\|x^k - P_C\left(x^k - 0.1 \cdot F(x^k)\right)\right\|$, where the mapping $F(x^k) \coloneqq Px^k + Qx^k + q$. From the common starting vector $x^0 = (1,1,1,1,1)^T$, under a 10-second CPU time limit, the algorithms REA1 and REA2 carry out 13282 and 12521 iterations, respectively. Figure 4.1 exhibits the result.

Example 4.2. To illustrate the asymptotic behavior of the continuous-time dynamical system (4) in an infinite-dimensional setting, we consider the Hilbert space $H \coloneqq \ell^2$ and a fixed positive integer $m \geq 2$. The feasible constraint subset $C \subset \ell^2$ is given by

$$C = \{x \in \ell^2 : x_1 = x_2 = \cdots = x_m = 0\}.$$

We consider the operator $B: \ell^2 \to \ell^2$ such that $B(x) = (\varphi_1, \varphi_2, \dots)$ where $\varphi_1 = \cdots = \varphi_m = 0$, $\varphi_{m+1} = x_{m+2}$, $\varphi_{m+2} = -x_{m+1}$ and $\varphi_j = 0$ for all $j \geq m+3$. Since $\langle Bx, x\rangle = 0$ for every x, it is a monotone and Lipschitz continuous operator. The associated structural and regularizing bifunctions are selected as $f(x,y) = \langle B(x), y - x\rangle$ and $g(x,y) = \langle x, y - x\rangle$, respectively. Under this setting, the variational inequality problem:

$$\text{Find } x^* \in C \text{ such that } \langle B(x^*), y - x^*\rangle \geq 0 \quad \forall\, y \in C, \qquad \left(\text{VIP}(B,C)\right)$$

is exactly $\text{EP}(f,C)$. It is obvious that $Sol(f,C) \neq \emptyset$ and $Sol\left(g, Sol(f,C)\right) = \{x^\dagger\}$, where $x^\dagger = 0$. The control parameters are set to $\alpha(t) = (1+t)^{-0.25}$ and $\lambda(t) = (1+t)^{-0.25}$. For an arbitrary initial vector x^0, by evaluating the system (4) through the projection P_C into C, we get the following linear differential system:

$$\begin{cases} \dot{x}_i(t) = -x_i(t), \ \ i = 1, \dots, m, \\ \dot{x}_{m+1}(t) = [-2\lambda(t)\alpha(t) + \lambda^2(t)\alpha^2(t)]x_{m+1}(t) - [2\lambda(t) - 2\lambda^2(t)\alpha(t)]x_{m+2}(t), \\ \dot{x}_{m+2}(t) = [-2\lambda(t)\alpha(t) + \lambda^2(t)\alpha^2(t)]x_{m+2}(t) + [2\lambda(t) - 2\lambda^2(t)\alpha(t)]x_{m+1}(t), \\ \dot{x}_j(t) = [-2\lambda(t)\alpha(t) + \lambda^2(t)\alpha^2(t)]x_j(t), \ \ \forall\, j \geq m+3, \\ x(0) = x^0. \end{cases}$$

Let $V(t) \coloneqq x_{m+1}^2(t) + x_{m+2}^2(t)$. Solving the last system, we receive

$$\begin{cases} x_i(t) = x_i(0)e^{-t} = x_i^0 e^{-t}, \ \ i = 1, \dots, m, \\ V(t) = V(0)e^{2\int_0^t [-2\lambda(s)\alpha(s) + \lambda^2(s)\alpha^2(s)]ds}, \\ x_j(t) = x_j^0 e^{\int_0^t [-2\lambda(s)\alpha(s) + \lambda^2(s)\alpha^2(s)]ds}, \ \ \forall\, j \geq m+3. \end{cases}$$

Let $\Gamma(t) \coloneqq -2\lambda(t)\alpha(t) + \lambda^2(t)\alpha^2(t)$, we have

$$\left\|x(t) - x^\dagger\right\|^2 = \sum_{i=1}^{m} \left|x_i^0\right|^2 e^{-2t} + V(0)e^{2\int_0^t \Gamma(s)ds} + \left(\|x^0\|^2 - \sum_{j=1}^{m+2} \left|x_j^0\right|^2\right) e^{2\int_0^t \Gamma(s)ds}.$$

By direct calculation, we have $\int_0^{+\infty} \Gamma(t)dt = \int_0^{+\infty} (-2(1+t)^{-0.75} + (1+t)^{-1.5})dt = -\infty$. Thus, we see that $\left\|x(t) - x^\dagger\right\|^2 \to 0$ as $t \to +\infty$, which validates strong convergence.

5. CONCLUSION

In this work, we have introduced and systematically analyzed a novel regularized extragradient-type dynamical system for solving monotone equilibrium problems. Under

appropriate conditions, the generated continuous trajectories are established to converge strongly to a unique solution of the bilevel equilibrium problem. Furthermore, by employing a forward Euler discretization scheme, we have developed two associated discrete iterative algorithms with relaxation parameters. The strong convergence of these algorithms can be proved by discrete-time analytic tools.

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