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INDUCED INNER PRODUCT STRUCTURES AND CAUCHY–SCHWARZ INEQUALITIES FOR LINEAR FUNCTIONALS

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Abstract. Linear functionals on finite-dimensional polynomial spaces generate fundamental algebraic and analytic structures, including moment sequences, moment matrices, and functional inequalities. Associated with a linear functional on polynomials of bounded degree is a moment matrix whose entries are given by the values of the functional on products of monomials and naturally exhibit a Hankel structure. Adopting an intrinsic functional-analytic viewpoint, this paper studies linear functionals on polynomial spaces without invoking any external representation framework. We develop a unified algebraic setting in which linearity, positivity-type conditions, moment matrices, and a functional inequality are examined simultaneously. We distinguish properties arising purely from linearity from those requiring additional structural assumptions. In particular, we establish a Cauchy-type inequality for linear functionals under mild algebraic conditions, independent of any a priori inner product structure. Under stronger positivity assumptions, the linear functional induces an inner product on polynomial spaces, with the associated moment matrix reflecting this structure precisely. Moreover, the Hankel structure of moment matrices is clarified as an intrinsic consequence of polynomial multiplication.

Keywords: linear functionals, inner product, moment matrix, moment sequence, Cauchy-Schwarz inequality.

1. INTRODUCTION

Linear functionals on polynomial spaces are fundamental objects in functional analysis, providing a natural framework for the study of duality, positivity, and induced algebraic structures [1,2]. When restricted to spaces of polynomials of bounded degree, they give rise to moment sequences and moment matrices whose structure reflects both algebraic and positivity properties.

The moment matrix associated with a linear functional is defined by the values of the functional on products of monomials and exhibits a characteristic Hankel structure, encoding invariance under polynomial multiplication and degree shifts [3,4]. Such matrices play a central role in the analysis of positivity conditions and have been extensively studied in both classical and truncated settings [5,6].

Another key aspect concerns inequalities associated with linear functionals. While classical inequalities such as the Cauchy's inequality are typically formulated in inner product spaces, it is natural to ask whether analogous inequalities can be derived intrinsically from properties of a linear functional. In this paper, we show that a Cauchy-type inequality can be established on polynomial spaces of bounded degree under mild algebraic conditions, without assuming an underlying inner product.

Under additional positivity assumptions, this framework naturally leads to inner product structures on polynomial spaces, thereby connecting linear functionals, functional inequalities, and moment matrices within a unified functional-analytic setting [2,7]. Recent developments in polynomial optimization and semidefinite programming further motivate such an intrinsic approach, as truncated moment matrices arise as fundamental algebraic objects independent of representability [8,9].

The aim of this paper is to clarify the structural relationships between linear functionals, induced inner products, and moment matrices on spaces of polynomials of bounded degree, emphasizing the distinction between properties following from linearity alone and those requiring positivity assumptions.

2. PRELIMINARIES AND NOTATION

Throughout this paper, all vector spaces are over the real field $\mathbb{R}$.

Let $\mathbb{P}_d$ represent the set of $n-$ variables real polynomials whose degree does not exceed d and by $\mathbb{M}_n(\mathbb{R})$ the algebra of real $n \times n$ matrices.

Let $\mathbb{N}$ be the set of non-negative integer. For $i = (i_1, i_2, \dots, i_n) \in \mathbb{N}^n,\ x = (x_1, \dots, x_n)$ we denote $x^i = x_1^{i_1}.x_2^{i_2} \dots x_n^{i_n},\ |i| = i_1 + i_2 + \dots + i_n$. Let $f \in \mathbb{P}_d, f$ can be written as

$$f = \sum_{|i| \le d} f_i x^i, \quad f_i \in \mathbb{R}. \tag{1}$$

2.1. Linear functionals

A linear functional on $\mathbb{P}_d$ is a mapping $L: \mathbb{P}_d \to \mathbb{R}$ satisfying

$$L(\alpha f + \beta g) = \alpha L(f) + \beta L(g), \qquad \alpha, \beta \in \mathbb{R}, f, g \in \mathbb{P}_d, \qquad [10, \text{p.} 51, \text{p.} 106]. \tag{2}$$

The set of all linear functionals on $\mathbb{P}_d$ forms its algebraic dual, denoted by $\mathbb{P}_d^*$, which serves as the foundational object in moment theory.

2.2. Moment sequences and moment matrices

Let us consider a linear functional L acting on the space $\mathbb{P}_d$, the associated moment sequence $\{s_k\}_{|k|\leq d}$ is defined by

$$s_k := L(x^k), \quad [11, \text{p}. 2]. \tag{3}$$

By [11], the corresponding moment matrix is the matrix $M = (s_{ij}) = (s_{i+j}), 0 \leq |i|, |j| \leq d$.

Such matrices have a characteristic Hankel structure [11, p. 2, p. 50].

2.3. Quadratic forms and positive (semi)definite matrices

The quadratic form of a symmetric matrix $M \in \mathbb{M}_n(\mathbb{R})$ is given by

$$q_M(x) = x^T M x, \ x \in \mathbb{R}^n, \ [12, \text{p}. 349]. \tag{4}$$

If $x^T M x \geq 0, \ \forall x \in \mathbb{R}^n$ then we say that M is positive semidefinite ($M \succcurlyeq 0$). If $x^T M x > 0$, $\forall x \in \mathbb{R}^n \backslash \{0\}, M$ is positive definite ($M \succ 0$) $[12, \text{p}. 340, \text{p}. 356]$.

2.4. Inner products and Cauchy–Schwarz inequality

Let V be a real vector space. We say that $B: V \times V \to \mathbb{R}$ constitutes a bilinear form provided it satisfies the linearity condition for each of its arguments. [13, p. 2].

On a real vector space V, a semi–inner product is a bilinear form $\langle \cdot, \cdot \rangle: V \times V \to \mathbb{R}$ satisfying:

1. Symmetric: $\langle f, g \rangle = \langle g, f \rangle$ for all $f, g \in V$;
2. Positive semidefinite: $\langle f, f \rangle \geq 0$ for all $f \in V$, [13, p. 5, p. 7].

A semi-inner product satisfying $\langle f, f \rangle = 0$ precisely when f is the zero polynomial is an inner product. Every inner product induces a quadratic form $q(f) = \langle f, f \rangle$, and conversely, positive definite quadratic forms give rise to inner products via polarization [12, p. 340].
Moreover, any inner product satisfies the Cauchy–Schwarz inequality [2, p. 3] satisfying

$$|\langle f, g \rangle|^2 \leq \langle f, f \rangle \langle g, g \rangle, \qquad f, g \in V. \tag{5}$$

Bilinear forms and inner products provide the natural algebraic framework for matrix representations of quadratic forms and in the analysis of moment matrices.

With the above notation and basic concepts in place, we now turn to the main results of the paper. The definitions introduced in Section 2 provide a unified framework for linear functionals, moment matrices, positivity, and inner product structures, which will be used throughout the subsequent analysis.

3. MAIN RESULTS

This section is devoted to the presentation and discussion of core theoretical results. Our analysis focuses on moment matrices associated with linear functionals on $\mathbb{P}_{2d}$ and examines how positivity and inner product structures on polynomial spaces are reflected in their matrix properties. The results are formulated using the framework established in Section 2.

3.1. Positivity of Linear Functionals and Induced Inner Products

Proposition 1. *Let L be a linear functional defined on $\mathbb{P}_{2d}$. The equivalence of the subsequent conditions can be established.*

(1) $\langle f, g\rangle := L(fg)$ defines an inner product on $\mathbb{P}_d$.

(2) $L(h^2) > 0$ for every nonzero polynomial $h \in \mathbb{P}_d$.

(3) The sequence $\{m_i = L(x^i)\}_{|i|\le 2d}$ is positive definite, i.e., the matrix $M = (m_{ij})$ is positive definite, where $m_{ij} = m_{i+j}$ for $|i|, |j| \le d$.

Proof.

$(1 \Rightarrow 2)$. Suppose $\langle f, g\rangle := L(fg)$ is an inner product on $\mathbb{P}_d$.

Given $h = \sum_{|i|\le d} h_i x^i, h \neq 0, h \in \mathbb{P}_d$. So $L(h^2) = \langle h, h\rangle > 0\ \forall h \neq 0$.

$(2 \Rightarrow 3)$. Suppose $L(h^2) > 0\ \forall h \neq 0, h \in \mathbb{P}_d$. Let

$$h = \sum_{|i|\le d} h_i x^i, h \neq 0 \Longrightarrow h^2 = \sum_{|i|,|j|\le d} h_i h_j x^{i+j}. \tag{6}$$

We obtain $L(h^2) = \sum_{|i|,|j|\le d} h_i h_j L(x^{i+j}) = \sum_{|i|,|j|\le d} h_i h_j m_{ij} > 0.$

We have $M = (m_{ij})_{|i|,|j|\le d} \in \mathbb{M}_N(\mathbb{R}),\ h = (h_i)_{|i|\le d} \in \mathbb{R}^N, h \neq 0,\ N = \binom{d+n}{n}$. Thus, $h^T S h = \sum_{|i|,|j|\le d} h_i h_j m_{ij} > 0$.

This leads to $M \succ 0$, which means the sequence $\{m_i = L(x^i)\}_{|i|\le 2d}$ is positive definite.

$(3 \Rightarrow 1)$. Suppose the matrix $M = (m_{ij})$ is positive definite, where $m_{ij} = m_{i+j} = L(x^{i+j})$ for $|i|, |j| \le d$. Let $\langle f, g\rangle := L(fg)\ \ \forall f, g \in \mathbb{P}_d$. Standard results in functional analysis [1] imply that a bilinear form induces an inner product when symmetry and positivity are both satisfied, i.e.

$$\langle h, h\rangle > 0\ \ \forall\ h \in \mathbb{P}_d \backslash \{0\}. \tag{7}$$

The bilinear form $\langle f, g\rangle = L(fg)$ is symmetric, since polynomial multiplication is commutative.

Given that $h = \sum_{|i|\le d} h_i x^i$. Since $M \succ 0$, this leads to

$$\langle h, h\rangle = L(h^2) = \sum_{|i|,|j|\le d} h_i h_j L(x^{i+j}) = \sum_{|i|,|j|\le d} h_i h_j m_{ij} = h^T M h > 0\ \ \forall h \in \mathbb{P}_d \backslash \{0\}, \tag{8}$$

and $\langle h, h\rangle = 0 \Leftrightarrow h^T M h = 0 \Leftrightarrow h$ is the zero polynomial.

Therefore, $\langle f, g\rangle = L(fg)$ defines an inner product on $\mathbb{P}_d$. □

Proposition 2. *Suppose that $\langle\ ,\ \rangle$ is an inner product on $\mathbb{P}_d$. The following statements are equivalent.*

(1) There exists a linear functional L on $\mathbb{P}_{2d}$ such that

$$L(fg) := \langle f, g\rangle, \tag{9}$$

and $L(f^2) > 0\ \forall f \neq 0$.

(2) $\langle x^i, x^j\rangle = \langle x^s, x^t\rangle$ *whenever* $i + j = s + t$ *for* $|i|, |j|, |s|, |t| \leq d$.

Proof.

$(1 \Rightarrow 2)$. Let L be a linear functional on $\mathbb{P}_{2d}$ such that $\langle f, g\rangle = L(fg)\ \forall f, g \in \mathbb{P}_d$.

We have $\langle x^i, x^j\rangle = L(x^i x^j) = L(x^{i+j}), \quad 0 \leq |i|, |j| \leq d;$

$$\langle x^s, x^t\rangle = L(x^s x^t) = L(x^{s+t}), \quad 0 \leq |s|, |t| \leq d.$$

If $i + j = s + t$ then $L(x^{i+j}) = L(x^{s+t})$. Thus, $\langle x^i, x^j\rangle = \langle x^s, x^t\rangle$ whenever $i + j = s + t$ for $|i|, |j|, |s|, |t| \leq d$.

$(2 \Rightarrow 1)$. Define $L(x^k) := \langle x^i, x^j\rangle$ for any $|i|, |j| \leq d$ with $i + j = k$.

Assumption (2) ensures that this definition is well defined. It follows from the linearity inherent in the inner product that L constitutes a linear functional. For arbitrary polynomials $f = \sum_{|i|\leq d} f_i x^i$, $g = \sum_{|j|\leq d} g_j x^j \in \mathbb{P}_d$, one has

$$L(fg) = \sum_{|i|,|j|\leq d} f_i g_j L(x^{i+j}) = \sum_{|i|,|j|\leq d} f_i g_j \langle x^i, x^j\rangle = \langle \sum_{|i|\leq d} f_i x^i, \sum_{|j|\leq d} g_j x^j\rangle = \langle f, g\rangle, \tag{10}$$

and $L(f^2) = \langle f, f\rangle > 0, \quad \forall f \neq 0.$ (11)

□

Remark 1. The bilinear form $\langle \cdot, \cdot \rangle$ defined by L is a semi-inner product if $L(p^2) \geq 0$ for all $p \in \mathbb{P}_d$.

The condition in Proposition 2 implies that the bilinear form depends only on the sum of multi-indices, which leads naturally to the moment matrix structure in Proposition 3.

Proposition 3. *Given* $M := (m_{ij}) \in \mathbb{M}_N(\mathbb{R})$, *where* $N = \binom{d+n}{n}$. *M represents the moment matrix associated with the linear functional* $L: \mathbb{P}_{2d} \longrightarrow \mathbb{R}$ *exactly when if* $m_{ij} = m_{st}$ *whenever* $i + j = s + t$, $|i|, |j|, |s|, |t| \leq d$. *Moreover, M is positive (semi)definite if and only if* $L(h^2) > 0\ \forall h \in \mathbb{P}_d \backslash \{0\}$ *(respectively,* $L(h^2) \geq 0\ \forall h \in \mathbb{P}_d$*).*

Proof.

$\Longrightarrow$ Suppose M is the moment matrix of linear functional $L: \mathbb{P}_{2d} \longrightarrow \mathbb{R}$. Define $m_k := L(x^k)$, $|k| \leq 2d$.

Then $M = (m_{ij}) = \left(L(x^{i+j})\right) = (m_{i+j})$, $|i|, |j| \leq d$. If $i + j = s + t$ then $L(x^{i+j}) = L(x^{s+t})\ \forall\ |i|, |j|, |s|, |t| \leq d$.

Thus, $m_{ij} = m_{st}\ \forall\ i + j = s + t,\ |i|, |j|, |s|, |t| \leq d$.

$\Longleftarrow$ Suppose $M := (m_{ij}) \in \mathbb{M}_N(\mathbb{R})$ s.t $m_{ij} = m_{st}\ \forall i + j = s + t, |i|, |j|, |s|, |t| \leq d$.

Then there exists a function $k \mapsto m_k$ such that $m_{ij} = m_{i+j}$. Let $L: \mathbb{P}_{2d} \longrightarrow \mathbb{R}$ be a linear functional such that $L(x^k) := m_k\ \forall\ |k| \leq 2d$. Then

$$h = \sum_{|i|\le 2d} h_i x^i, h \in \mathbb{P}_{2d} \Longrightarrow L(h) := \sum_{|i|\le 2d} h_i L(x^i) = \sum_{|i|\le 2d} h_i m_i. \tag{12}$$

Therefore, $\forall\ |i|, |j| \le d: L(x^{i+j}) = m_{i+j} = m_{ij}$. M will be the moment matrix of linear functional $L: \mathbb{P}_{2d} \longrightarrow \mathbb{R}$. Moreover, when M is the moment matrix of L. Let

$$h = \sum_{|i|\le d} h_i x^i \in \mathbb{P}_d, \qquad h = (h_i)_{|i|\le d} \in \mathbb{R}^N. \tag{13}$$

We obtain

$$L(h^2) = L\left(\sum_{|i|,|j|\le d} h_i h_j x^{i+j}\right) = \sum_{|i|,|j|\le d} h_i h_j L(x^{i+j}) = \sum_{|i|,|j|\le d} h_i h_j m_{ij} = h^T M h\,. \tag{14}$$

This implies $L(h^2) > 0\ \forall h \in \mathbb{P}_d \backslash \{0\} \Leftrightarrow h^T M h > 0$, i.e., $M \succ 0$.

The semidefinite case is analogous. □

Remark 2. The moment matrix appearing in Propositions 1–3 is of Gram type [3,7,11].

3.2. A Cauchy–Schwarz Inequality for Linear Functionals

Theorem 1. *Consider a linear functional L on $\mathbb{P}_{2d}$ whose moment matrix is positive semidefinite. Then the following inequality holds for all $f, g \in \mathbb{P}_d$:*

$$\big(L(fg)\big)^2 \le L(f^2)L(g^2). \tag{15}$$

Moreover, if f, g are linearly dependent then $\big(L(fg)\big)^2 = L(f^2)L(g^2)$.

Proof.

From Proposition 3, we have $L(h^2) \ge 0, \forall h \in \mathbb{P}_d$. Then $0 \le L((f - \alpha g)^2),\ \ \forall \alpha \in \mathbb{R}$.

So $0 \le L(f^2) - 2\alpha L(fg) + \alpha^2 L(g^2),\ \forall \alpha \in \mathbb{R}$. Thus, $\Delta' = \big(L(fg)\big)^2 - L(f^2)L(g^2) \le 0 \Leftrightarrow \big(L(fg)\big)^2 \le L(f^2)L(g^2)$.

Suppose that $f = \alpha g$ for some $\alpha \in \mathbb{R}$ then

$$\big(L(fg)\big)^2 = \alpha^2\big(L(f^2)\big)^2, \text{ and } L(f^2)L(g^2) = \alpha^2 L(f^2)L(f^2) = \alpha^2\big(L(f^2)\big)^2. \tag{16}$$

Therefore, $\big(L(fg)\big)^2 = L(f^2)L(g^2)$. □

Theorem 2. *Consider a linear functional L on $\mathbb{P}_{2d}$. If u, v are linearly independent in $\mathbb{P}_d$ and $L(h^2) \ge 0$ for all $h \in$ span$\{u, v\}$ then*

$$\big(L(uv)\big)^2 \le L(u^2)L(v^2). \tag{17}$$

Moreover, if $L(u^2) = 0$ and $L(v^2) = 0$, then $L(uv) = 0$.

Proof.

$L(h^2) \geq 0\ \forall h \in \text{span}\{u,v\}$ and u,v are linearly independent. Then $0 \leq L((u-\alpha v)^2)$, for all $\alpha \in \mathbb{R}$.

One can arrive at this conclusion by applying a methodology comparable to that employed in the verification of Theorem 1.

Suppose $L(u^2)=0$ and $L(v^2)=0$ then

$$0 \leq L((u+v)^2) = 2L(uv) \text{ and } 0 \leq L((u-v)^2) = -2L(uv). \tag{18}$$

Therefore, $L(uv)=0$, i.e., $\left(L(uv)\right)^2 = L(u^2)L(v^2)$. □

Example 1. Let $\mathbb{P}_2$ be the set of one – variable polynomials of degree up to 2 and $L\colon \mathbb{P}_2 \to \mathbb{R}$ be defined by letting $L(p)$ be the coefficient of x^2 in the polynomial p. This is a well-defined linear functional on $\mathbb{P}_2$.

Let $u=1$ and $v=x$. Then $\text{span}\{u,v\} = \{h = a+bx \mid a,b \in \mathbb{R}\} \subset \mathbb{P}_2$. For such a function h,

$$h^2 = a^2 + 2abx + b^2x^2. \tag{19}$$

Therefore, $L(h^2) = b^2 \geq 0,\ \ \forall h \in \text{span}\{1,x\}$. Hence, $L(u^2)=0$, $L(v^2)=1$ and $\left(L(uv)\right)^2 = 0$.

These observations imply that $\left(L(uv)\right)^2 \leq L(u^2)L(v^2)$, and hence the Cauchy–Schwarz inequality holds for the pair (u,v). □

The assumption $h \in \text{span}\{u,v\}$ in Theorem 2 is essential. The following example demonstrates its necessity.

Example 2. Let $\mathbb{P}_4$ be the set of one – variable polynomials of degree up to 4 and $L\colon \mathbb{P}_4 \to \mathbb{R}$ be defined by

$$L(1)=1, \qquad L(x)=0, \qquad L(x^2)=0, \qquad L(x^3)=1, \qquad L(x^4)=1. \tag{20}$$

This is a well-defined linear functional on $\mathbb{P}_4$. Let $f=1$ and $g=x^2$. Then $\text{span}\{f,g\} = \{h = a+bx^2 \mid a,b \in \mathbb{R}\} \subset \mathbb{P}_4$. For such a function h,

$$h^2 = a^2 + 2abx^2 + b^2x^4. \tag{21}$$

Therefore, $L(h^2) = a^2+b^2 \geq 0,\ \forall h \in \text{span}\{f,g\}$. Take $u = x+x^2$ and $v = x-x^2$, then $L(u^2)=3$, $L(v^2)=-1$ and $L(uv)=-1$. Thus, $\left(L(uv)\right)^2 = 1$ and $L(u^2)L(v^2) = -3$. So the Cauchy–Schwarz inequality fails for the pair (u,v). □

Example 3. Let $\mathbb{P}_2$ be the set of two – variable polynomials of degree up to 2 and $L\colon \mathbb{P}_2 \to \mathbb{R}$ be defined by letting $L(p)$ be the coefficient of x^2 in the polynomial p.

Let $f=1, g=x+y$. Then for any $h = a+b(x+y) \in \text{span}\{f,g\}$, one easily checks that $L(h^2)=b^2 \geq 0,\ \forall h \in \text{span}\{f,g\}$.

In particular, $L(f^2)=0$, $L(g^2)=1$, and $\left(L(fg)\right)^2=0$, so $\left(L(fg)\right)^2 \leq L(f^2)L(g^2)$. Thus, the Cauchy–Schwarz inequality holds for (f,g).

□

3.3. Some Applications

In this subsection, we discuss some applications of Theorem 1 and 2 to proving elementary inequalities. In addition, Proposition 1 allows us to design a Python algorithm for verifying whether a linear functional is an inner product or a semi–inner product.

Example 4. Let $c \geq 0$. Prove that $(a+2bc)^2 \leq (1+4c)(a^2+cb^2)$, for $a, b \in \mathbb{R}$.

Proof.

Let $\mathbb{P}_2$ be the set of one – variable polynomials of degree up to 2 and $L: \mathbb{P}_2 \to \mathbb{R}$ be defined by $L(1)=1,\ L(x)=0,\ L(x^2)=c$. Then $\langle f, g \rangle := L(fg)$ for $f, g \in \mathbb{P}_1$ is a semi-inner product on $\mathbb{P}_1$.

Let $f = a + bx,\ g = 1 + 2x$. We obtain

$$L(fg) = a + 2bc, \qquad L(f^2) = a^2 + cb^2, \qquad L(g^2) = 1 + 4c. \tag{22}$$

By Theorem 1 we have $(a+2bc)^2 \leq (1+4c)(a^2+cb^2)$ for all $a, b, c \in \mathbb{R}, c \geq 0$.

□

Example 5. Let $c \geq 1$. Prove that

$$(abc - 3a - 2b + 6)^2 \leq (ca^2 - 4a + 4)(cb^2 - 6b + 9), \qquad a, b \in \mathbb{R}, 3a \neq 2b. \tag{23}$$

Proof.

Let $\mathbb{P}_4$ be the set of one – variable polynomials of degree up to 4 and $L: \mathbb{P}_4 \to \mathbb{R}$ satisfies

$$L(1) = 1, \quad L(x) = 0, \quad L(x^2) = -1, \quad L(x^3) = 0, \quad L(x^4) = c. \tag{24}$$

Let $h \in \operatorname{span}\{1, x^2\}$ then $L(h^2) = (\alpha - \beta)^2 + (c-1)\beta^2 \geq 0,\ \forall c \geq 1, \alpha, \beta \in \mathbb{R}$.

Let $f = 2 + ax^2,\ g = 3 + bx^2$, then $\operatorname{span}\{f, g\} \subset \operatorname{span}\{1, x^2\}$. This implies that f, g are linearly independent (for $3a \neq 2b$) and $L(h^2) \geq 0$ for all $h \in \operatorname{span}\{f, g\}$. We have

$$L(fg) = abc - 3a - 2b + 6, L(f^2) = ca^2 - 4a + 4, L(g^2) = cb^2 - 6b + 9. \tag{25}$$

By Theorem 2 we have $(abc - 3a - 2b + 6)^2 \leq (ca^2 - 4a + 4)(cb^2 - 6b + 9)$ for all $a, b, c \in \mathbb{R},\ 3a \neq 2b, c \geq 1$. □

Example 6. Let $c \geq 2$. Prove that

$$(abc - 6a - 4b + 12)^2 \leq (ca^2 - 8a + 8)(cb^2 - 12b + 18), a, b \in \mathbb{R},\ 3a \neq 2b. \tag{26}$$

Proof.

Let $\mathbb{P}_4$ be the set of bivariate polynomials of degree up to 4 and $L: \mathbb{P}_4 \to \mathbb{R}$ satisfies

$$\begin{gathered} L(1) = 1, \quad L(x) = 0, \quad L(y) = 0, \quad L(x^2) = -1, \quad L(y^2) = -1, \\ L(xy) = 0, \quad L(x^3) = 0, \quad L(x^2y) = 0, \quad L(xy^2) = 0, \quad L(y^3) = 0, \\ L(x^4) = c, \quad L(y^4) = c, \quad L(x^2y^2) = 0, \quad L(x^3y) = 0, \quad L(xy^3) = 0. \end{gathered} \tag{27}$$

Let $h \in \text{span}\{1, x^2+y^2\}$ then $L(h^2) = (\alpha - 2\beta)^2 + 2(c-2)\beta^2 \geq 0,\ \forall c \geq 2, \alpha, \beta \in \mathbb{R}$.

Let $f = 4 + a(x^2+y^2)$, $g = 6 + b(x^2+y^2)$, then $\text{span}\{f, g\} \subset \text{span}\{1, x^2+y^2\}$. This implies that f, g are linearly independent (for $3a \neq 2b$) and $L(h^2) \geq 0$ for all $h \in \text{span}\{f, g\}$.

We obtain $L(fg) = 2(abc - 6a - 4b + 12)$, $L(f^2) = 2(ca^2 - 8a + 8)$ and $L(g^2) = 2(cb^2 - 12b + 18)$.

By Theorem 2 we have $(abc - 6a - 4b + 12)^2 \leq (ca^2 - 8a + 8)(cb^2 - 12b + 18)$,

$a, b, c \in R,\ 3a \neq 2b, c \geq 2$. □

Based on Proposition 1, Algorithm 1 provides a criterion for determining whether a linear functional induces an inner or semi-inner product.

Objective.
To determine whether a linear functional L induces an inner product, a semi-inner product, or neither on $\mathbb{P}_d$.

Input.

A positive integer n and a nonnegative integer d.
A linear functional $L: \mathbb{P}_{2d} \longrightarrow \mathbb{R}$, specified by its moments $s_\alpha \coloneqq L(x^\alpha)$, $|\alpha| \leq 2d$, where $\alpha \in \mathbb{N}^n, x^\alpha = x_1^{\alpha_1} \dots x_n^{\alpha_n}$.

Output.

A characterization of the bilinear form $\langle u, v\rangle \coloneqq L(uv)$, $\quad u, v \in \mathbb{P}_d$, as an inner product, a semi-inner product, or neither.

Algorithm 1.

1. Define $\Lambda_d = \{\alpha \in \mathbb{N}^n : |\alpha| \leq d\}$.
2. Arrange Λ_d in graded lexicographical order $\alpha^{(0)}, \alpha^{(1)}, \dots, \alpha^{(N-1)}, N = \binom{d+n}{n}$.
3. Construct the moment matrix $M \in \mathbb{M}_N(\mathbb{R})$:
$$M_{ij} = L(x^{\alpha^{(i)}+\alpha^{(j)}}) = s_{\alpha^{(i)}+\alpha^{(j)}}. \tag{28}$$
4. Compute eigenvalues $\sigma(M) = \{\lambda_0, \dots, \lambda_{N-1}\}$.
5. If $\lambda_i > 0$ for all i, then L induces an inner product.
6. If $\lambda_i \geq 0$ for all $i$and $\lambda_j = 0$ for some j, then L induces a semi-inner product.
7. Otherwise, it does not induce a (semi-)inner product.

The algorithm was implemented in Python in order to validate the theoretical results and to generate numerical examples.

A Python implementation of Algorithm 1 is provided in Appendix A.

3.4. Cauchy–Schwarz Inequality for a System of k Vectors

Let V be an inner product space over $\mathbb{R}$, and let $u, v \in V$. The classical Cauchy–Schwarz inequality asserts that

$$|\langle u, v\rangle|^2 \leq \langle u, u\rangle\langle v, v\rangle, \tag{29}$$

where $\langle \cdot , \cdot \rangle$ denotes the inner product on V. Moreover, this inequality is equivalent to the non-negativity of the Gram determinant associated with u and v:

$$\det\begin{pmatrix} \langle u, u\rangle & \langle u, v\rangle \\ \langle v, u\rangle & \langle v, v\rangle \end{pmatrix} \geq 0. \tag{30}$$

Hence, the case of two vectors can be naturally extended to a system of k vectors.

Theorem 3. *Consider a linear functional L on $\mathbb{P}_{2d}$. If its corresponding moment matrix is positive semidefinite, then*

$$\det\begin{pmatrix} L(f_1^2) & \cdots & L(f_1 f_k) \\ \vdots & \ddots & \vdots \\ L(f_k f_1) & \cdots & L(f_k^2) \end{pmatrix} \geq 0 \qquad \forall\, f_1, \ldots, f_k \in \mathbb{P}_d. \tag{31}$$

This result is well known in linear algebra and inner product space theory. A natural open question is whether Theorems 1 and 2 admit extensions to systems of k polynomial.

4. CONCLUSION

We have investigated moment-related structures from an intrinsic perspective based on linear functionals on finite-dimensional polynomial spaces. Starting from a linear functional on $\mathbb{P}_{2d}$, we analyzed the associated moment sequences and moment matrices, identifying the Hankel structure as a direct consequence of polynomial multiplication.

We established a Cauchy-type inequality under basic algebraic conditions, without assuming an underlying inner product structure. Under stronger positivity assumptions, the same framework yields inner product structures, with moment matrices serving as their matrix representations.

This approach provides a clear hierarchy between linearity, functional inequalities, and inner product structures, offering a coherent refinement of classical moment-theoretic ideas and a streamlined functional-analytic perspective.

APPENDIX A. Python Implementation of Algorithm 1

This appendix provides a Python implementation of Algorithm 1, which is used to verify the positivity properties of the moment matrix associated with a given linear functional.

The implementation is intended solely for numerical validation and illustrative purposes and does not form part of the theoretical contributions of this paper.

```
import numpy as np
from numpy.linalg import eigvalsh
from itertools import product
# A.1 Multi-index generation (graded lex order)
def multi_indices(n_vars, max_degree):
    indices = []
```

```
    for total_degree in range(max_degree + 1):
        for alpha in product(range(total_degree + 1), repeat=n_vars):
            if sum(alpha) == total_degree:
                indices.append(alpha)
    return indices
```

 # A.2 Moment matrix construction

```
def moment_matrix(n_vars, d, L_func):
    indices = multi_indices(n_vars, d)
    N = len(indices)
    M = np.zeros((N, N))

    for i in range(N):
        for j in range(N):
            alpha = indices[i]
            beta = indices[j]
            gamma = tuple(a + b for a, b in zip(alpha, beta))
            M[i, j] = L_func(gamma)

    # numerical symmetrization
    M = (M + M.T) / 2.0
    return M, indices
```

 # A.3 Eigenvalue test

```
def check_positive_definiteness(M, tol=1e-10):
    eigvals = eigvalsh(M)

    if np.all(eigvals > tol):
        return "positive_definite", eigvals

    if np.all(eigvals >= -tol) and np.any(np.abs(eigvals) <= tol):
        return "positive_semidefinite", eigvals

    return "indefinite", eigvals
```

 # A.4 Full test

```
def moment_matrix_test(n_vars, d, L_func):
    M, indices = moment_matrix(n_vars, d, L_func)
    status, eigvals = check_positive_definiteness(M)
    return M, eigvals, status
```

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