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STUDY OF STATIC BENDING OF FUNCTIONALLY GRADED BEAMS USING ANALYTICAL METHOD

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Abstract. Beams made from variable mechanical properties materials are increasingly used in the fields of construction and transportation. The article presents a study on the static bending response of functionally graded composite beams resting on a two-parameter elastic foundation based on an exact solution. The material of the plate varies exponentially with the thickness variation. The calculations are formulated based on Timoshenko's first-order shear deformation theory, and the equilibrium equations of the beam are derived using the principle of virtual work. An analytical method is employed to derive expressions for displacement and rotation at any point along the beam. The reliability of the study is validated by comparison with previously published solutions. Furthermore, this study also investigates the effects of material, geometric, and elastic foundation parameters on the displacement and rotation responses of the composite beam. This research serves as a significant foundation for engineers in designing and manufacturing practical structures.

Keywords: Static bending response; Functionally graded composite beams; Two-parameter elastic foundation; Timoshenko's first-order shear deformation theory; Principle of virtual work.

1. INTRODUCTION

Nowadays, to enable structures to withstand complex types of loads (shockwave loads, temperature variations, and so on), scientists have developed new materials with superior advantages in terms of load-bearing capacity. Examples include high-strength composite materials, graphene-reinforced composites, and composites made by blending ceramics and metals (functionally graded materials, FGMs). These materials can be used to manufacture bomb-resistant bunker doors, partitions for nuclear reactors, and more.

To assist engineers in designing products with the highest operational efficiency, extensive research is required across various aspects. Among these, the study of the mechanical responses of beams, plates, and shells made of functionally graded composite materials is an urgent requirement. Reddy [1] presented the static bending displacement and its variation over time under dynamic loads for functionally graded composite plates, based on both numerical simulations and exact solutions. Ninh [2] employed various beam theories to address the free vibration problem of multilayer beams with bi-directional functionally graded properties resting partially on an elastic foundation. Duc et al [3] utilized the finite element method to analyze the static buckling response of plates made of flexoeletric material. Lan et al [4] applied a novel approach to determine the stability responses of multilayered composite beams based on a new shear deformation theory. Nam et al. [5] analyzed the free and forced vibrations of shells with with Shear Connectors. Using Laplace transformations, Tho et al [6] clarified the relationship between tuned mass damper and beams and space frame systems. Yu et al [7] examined the stability of functionally graded composite plates with cracks under a thermal environment. Furthermore, the vibrational and stability responses of functionally graded composite beams have been investigated using analytical and Ritz methods in studies [8]-[9].

It is clear from previous research that finding out the displacement and rotation angles of functionally graded composite beams resting on a two-parameter elastic foundation is still an interesting topic that needs a lot more research, especially analytical methods. This is also the main purpose of this paper and the problem it aims to address.

2. CALCULATION FORMULAS

The beam subjected to a uniformly distributed load P_L has a calculation model as shown in Figure 1. The beam has a length of L and the height of the cross-section is h.

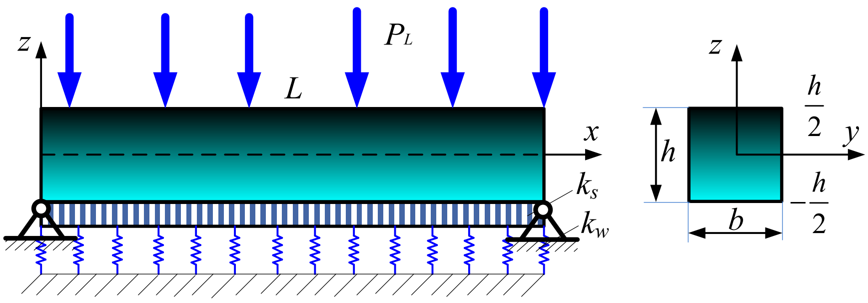


Figure 1. Model of a beam subjected to a uniformly distributed load resting on a two-parameter elastic foundation.

The composite beam is fabricated by blending ceramic and metal, where the volume fraction of ceramic is V_c, and the volume fraction of metal is V_m, These two components are related as shown in [1], [7]:

$$V_c = \left(\frac{z}{h} + \frac{1}{2}\right)^n ; V_m = 1 - V_c \tag{1}$$

Where the parameter n is referred to as the material volume exponent, the subscript m represents metal, and the subscript c represents ceramic.

The modulus of elasticity and Poisson's ratio vary along the thickness of the beam according to the following functions [1], [7]:

$$\{E(\text{z});\ \nu_c(\text{z})\} = \{E_m;\ \nu_m\} + \left(\{E_c;\ \nu_c\} - \{E_m;\ \nu_m\}\right)\left(\frac{1}{2} + \frac{z}{h}\right)^n \tag{2}$$

Using Timoshenko's first-order shear deformation theory, the displacement field at any point on the beam with coordinates (x, z) is expressed as:

$$\begin{cases} u(x,z) = u_0(x,0) + z\beta_x \\ w(x,z) = w_0(x,0) \end{cases} \tag{3}$$

Where $u_0(x,0)$, $w_0(x,0)$ are the axial and transverse displacements at the mid-surface of the beam, respectively u and w are the axial and transverse displacements along the Ox và Oz axes, respectively, and β_x represents the rotation of the beam's cross-section.

The components of axial strain and shear strain in the beam are given as follows:

$$\begin{gathered} \varepsilon_{xx} = \frac{\partial u}{\partial x} = \frac{\partial u_0}{\partial x} + z\frac{\partial \beta_x}{\partial x} = \varepsilon_{0x} + z\varepsilon_{1x} \\ \gamma_{xz} = \beta_x + \frac{\partial w_0}{\partial x} \end{gathered} \tag{4}$$

The specific expressions for the strain components in the beam are as follows:

$$\varepsilon_{0x} = \frac{\partial u_0}{\partial x},\ \varepsilon_{1x} = \frac{\partial \beta_x}{\partial x} \tag{5}$$

The components of normal stress and shear stress at any point can be calculated using Hooke's law as follows:

$$\begin{cases} \sigma_{xx} = E(z)\varepsilon_{xx} \\ \tau_{xz} = \dfrac{E(z)}{2(1+\nu)}\gamma_{xz} \end{cases} \tag{6}$$

The virtual work of internal forces of the beam during deformation is expressed as:

$$\delta U = b\int_0^L \int_{-\frac{h}{2}}^{\frac{h}{2}} \left(\sigma_{xx}\delta\varepsilon_{xx} + \tau_{xz}\delta\gamma_{xz}\right) dzdx = b\int_0^L \int_{-\frac{h}{2}}^{\frac{h}{2}} \begin{pmatrix} \varepsilon_{0x}^T E(z)\delta\varepsilon_{0x} + \varepsilon_{0x}^T zE(z)\delta\varepsilon_{1x} \\ +\varepsilon_{1x}^T E(z) z\delta\varepsilon_{0x} + \varepsilon_{1x}^T E(z) z^2\delta\varepsilon_{1x} \end{pmatrix} dzdx$$

$$+b\int_0^L \int_{-\frac{h}{2}}^{\frac{h}{2}} \left(\gamma_{xz}\left(\frac{5}{6}\frac{E(z)}{2(1+\nu)}\right)\delta\gamma_{xz}\right) dzdx \tag{7}$$

$$= b\int_0^L \int_{\frac{-h}{2}}^{\frac{h}{2}} \begin{pmatrix} \frac{\partial^2 u_0}{\partial x^2} E(z)\delta u_0 + \frac{\partial^2 u_0}{\partial x^2} zE(z)\delta\beta_x \\ +\frac{\partial^2 \beta_x}{\partial x^2} zE(z)\delta u_0 + \frac{\partial^2 \beta_x}{\partial x^2} E(z) z^2\delta\beta_x \end{pmatrix} dzdx$$

$$+b\int_0^L \int_{\frac{-h}{2}}^{\frac{h}{2}} \left(\left(\beta_x + \frac{\partial w_0}{\partial x}\right)\left(\frac{E(z)}{2(1+\nu)}\right)\delta\beta_x + \left(\frac{\partial \beta_x}{\partial x} + \frac{\partial^2 w_0}{\partial x^2}\right)\left(\frac{5}{6}\frac{E(z)}{2(1+\nu)}\right)\delta w_0\right) dx$$

where b is the width of the beam (Figure 1). The virtual work done by the foundation on the beam can be expressed as:

$$\delta U_n = b\int_0^L \left(k_w w_0 \delta w_0 + k_s \frac{\partial w_0}{\partial x}\delta\frac{\partial w_0}{\partial x}\right) dx \tag{8}$$

where k_w and k_s are the stiffness parameters of the foundation. The virtual work of external forces acting on the beam is expressed as:

$$\delta W = b\int_L P_L \delta w_0 dx \tag{9}$$

where P_L is the external force applied to the beam.

For the beam to be in equilibrium, the total virtual work of internal forces and external forces must be equal:

$$\delta U + \delta U_n = \delta W \tag{10}$$

Substituting the expressions for virtual work (7)–(9) into (10), and then grouping the terms based on the independent operators δu_0, $\delta\beta_x$ and δw_0, we obtain the following system of three equilibrium equations for each variable:

$$\delta u_0 : \frac{\partial^2 u_0}{\partial x^2} \int_{\frac{-h}{2}}^{\frac{h}{2}} E(z) dz + \frac{\partial^2 \beta_x}{\partial x^2} \int_{\frac{-h}{2}}^{\frac{h}{2}} zE(z) dz = 0 \tag{11}$$

$$\delta\beta_x : \frac{\partial^2 u_0}{\partial x^2}\int_{\frac{-h}{2}}^{\frac{h}{2}} zE(z)\,dz + \frac{\partial^2 \beta_x}{\partial x^2}\int_{\frac{-h}{2}}^{\frac{h}{2}} z^2 E(z)\,dz + \left(\beta_x + \frac{\partial w_0}{\partial x}\right)\int_{\frac{-h}{2}}^{\frac{h}{2}} \frac{5E(z)}{12(1+\nu)}\,dz = 0 \tag{12}$$

$$\delta w_0 : \left(\frac{\partial \beta_x}{\partial x} + \frac{\partial^2 w_0}{\partial x^2}\right)\int_{\frac{-h}{2}}^{\frac{h}{2}} \frac{5Ez^2E(z)}{12(1+\nu)}\,dz + k_w w_0 + k_s\frac{\partial^2 w_0}{\partial x^2} = P_L \tag{13}$$

Define the coefficients as follows:

$$A = \int_{\frac{-h}{2}}^{\frac{h}{2}} E(z)\,dz;\; B = \int_{\frac{-h}{2}}^{\frac{h}{2}} zE(z)\,dz$$
$$C = \int_{\frac{-h}{2}}^{\frac{h}{2}} z^2E(z)\,dz;\; D = \int_{\frac{-h}{2}}^{\frac{h}{2}} \frac{5E(z)}{12(1+\nu(z))}\,dz \tag{14}$$

The equations are simplified as follows:

$$\delta u_0 : A\frac{\partial^2 u_0}{\partial x^2} + B\frac{\partial^2 \beta_x}{\partial x^2} = 0 \tag{15}$$

$$\delta\beta_x : B\frac{\partial^2 u_0}{\partial x^2} + C\frac{\partial^2 \beta_x}{\partial x^2} + D\left(\beta_x + \frac{\partial w_0}{\partial x}\right) = 0 \tag{16}$$

$$\delta w_0 : D\left(\frac{\partial \beta_x}{\partial x} + \frac{\partial^2 w_0}{\partial x^2}\right) + k_w w + k_s\frac{\partial^2 w_0}{\partial x^2} = P_L \tag{17}$$

From (15): $\frac{\partial^2 u_0}{\partial x^2} = -\frac{B}{A}\frac{\partial^2 \beta_x}{\partial x^2}$, substituting into (16): $\left(C - \frac{B^2}{A}\right)\frac{\partial^2 \beta_x}{\partial x^2} + D\left(\beta_x + \frac{\partial w_0}{\partial x}\right) = 0$, we obtain two equations with two unknowns:

$$\left(C - \frac{B^2}{A}\right)\frac{\partial^2 \beta_x}{\partial x^2} + D\left(\beta_x + \frac{\partial w_0}{\partial x}\right) = 0 \tag{18}$$

$$D\left(\frac{\partial \beta_x}{\partial x} + \frac{\partial^2 w_0}{\partial x^2}\right) + k_w w_0 + k_s\frac{\partial^2 w_0}{\partial x^2} = P_L \tag{19}$$

For the case of a simply supported beam at both ends (The displacements and bending moments at both ends of the beam are set to zero), the analytical solution has the form:

$$\begin{aligned}
\beta_x &= \sum_{m=1}^{\infty} B_\beta \cos\left(\frac{m\pi x}{L}\right) \\
w_0 &= \sum_{m=1}^{\infty} W_0 \sin\left(\frac{m\pi x}{L}\right) \\
P_L &= \sum_{m=1}^{\infty} \frac{4P_0}{m\pi} \sin\left(\frac{m\pi x}{L}\right)
\end{aligned} \tag{20}$$

Substituting (20) into (18)-(19), we obtain:

$$\left[D - \left(C - \frac{B^2}{A}\right)\left(\frac{m\pi}{L}\right)^2\right] B_\beta + D\frac{m\pi}{L} W_0 = 0 \tag{21}$$

$$-D\left(B_\beta \frac{m\pi}{L} + W_0\left(\frac{m\pi}{L}\right)^2\right) + k_w W_0 - k_s W_0\left(\frac{m\pi}{L}\right)^2 = \frac{4P_0}{m\pi} \tag{22}$$

From (21), the following can be derived:

$$B_\beta = \frac{D\frac{m\pi}{L}}{\left[\left(C - \frac{B^2}{A}\right)\left(\frac{m\pi}{L}\right)^2 - D\right]} W_0 \tag{23}$$

Substituting (23) into (22), the resulting equation is:

$$W_0 = \frac{4P_0}{m\pi} \frac{1}{\left(\frac{D^2\left(\frac{m\pi}{L}\right)^2}{\left[D - \left(C - \frac{B^2}{A}\right)\left(\frac{m\pi}{L}\right)^2\right]} - D\left(\frac{m\pi}{L}\right)^2 + k_w W_0 - k_s W_0\left(\frac{m\pi}{L}\right)^2\right)} \tag{24}$$

Finally, the expression for the displacement w_0 takes the form:

$$w_0 = \sum_{m=1}^{\infty} \frac{4}{m\pi} \left\{ \frac{P_0}{\left(\frac{D^2\left(\frac{m\pi}{L}\right)^2}{\left[D - \left(C - \frac{B^2}{A}\right)\left(\frac{m\pi}{L}\right)^2\right]} - D\left(\frac{m\pi}{L}\right)^2\right) + k_w - k_s\left(\frac{m\pi}{L}\right)^2} \right\} \sin\left(\frac{m\pi x}{L}\right) \tag{25}$$

From expression (24), the amplitude B_β can be easily determined using formula (23). Consequently, the rotation angle β_x can be calculated using formula (20).

3. VERIFICATION EXAMPLE

The first example compares the maximum deflection of a homogeneous beam with a length L = 10 m, thickness h (L/h=10 and 20), and cross-sectional width b, he material properties are E = 30 MPa, and Poisson's ratio ν = 0.3. The beam is subjected to a uniformly distributed load with an intensity q_0 = 1 N/m, the maximum deflection is calculated using the dimensionless formula: $w^*_{max} = 100 w_{0max} \dfrac{EI}{q_0 L^4}$. The maximum deflection at the mid-span of the beam is calculated and compared with results from previously published studies, as shown in Table 1. It can be easily observed that the results from this paper closely align with the published data, demonstrating the reliability of the proposed calculation theory. In Table 1, GT1 and GT2 denote the exact solutions based on classical beam theory and first-order beam theory, respectively, while FEM represents the solution obtained using the finite element method.

Table 1. Comparison of Maximum Deflection for a Simply Supported Beam under Uniformly Distributed Load.

L/h	GT1 [10]	GT1 [11]	GT2 [10]	GT2 [11]	FEM [12]	Bài báo
10	1.313	1.302	1.348	1.334	1.334	1.340
20	1.313	1.302	1.321	1.310	1.309	1.315
100	1.131	-	1.313	-	-	1.307

Table 2. Comparison of Maximum Deflection $w^* = \dfrac{EI}{q_0 L^4} w_{0\max}$ of a Simply Supported Beam.

Foundation parameters		L/h=120			
T_w	T_s	CP [13]	CX [13]	CX [14]	Bài báo
0	0	1.302	1.302	1.303	1.306
	10	0.644	0.644	0.645	0.649
	25	0.366	0.366	0.367	0.370
10	0	1.180	1.180	1.181	1.185
	10	0.613	0.613	0.614	0.617
	25	0.355	0.355	0.356	0.359
100	0	0.640	0.640	0.640	0.644
	10	0.425	0.425	0.426	0.430
	25	0.282	0.282	0.283	0.286

Next, the paper compares the deflection of a beam resting on a two-parameter elastic foundation. The beam has a length L, a cross-sectional width b and a height h, it is subjected to a uniformly distributed load of intensity P_0, and the elastic foundation is characterized by parameters $T_w = \dfrac{k_w L^4}{EI}$, $T_s = \dfrac{k_s L^2}{EI}$, the moment of inertia is given as $I = bh^3/12$. The comparison results are presented in Table 2, where reference [9] uses the collocation method (CP) and exact solution (CX), and reference [10] uses the exact solution.

4. NUMERICAL INVESTIGATION RESULTS

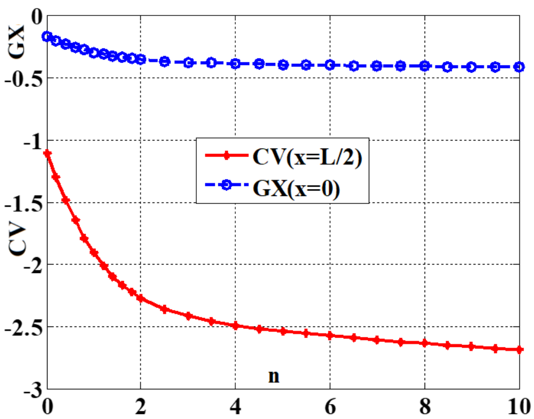
This section presents the results of the bending deflection calculations for a functionally graded composite beam made of ceramic and metal. The material parameters are as follows: E_c = 380 GPa, Em = 70 GPa, $\nu_c = \nu_m = 0.3$, The beam has a length L, and its height h is variable. he two elastic foundation parameters C_w and C_s are calculated using the following formulas: $C_w = \dfrac{12 k_w L^4}{E h_0^3}$, $C_s = \dfrac{12 k_s L^2}{E h_0^3}$. The beam is subjected to a uniformly distributed load with an intensity P_0, he displacement and rotation of the beam are calculated using the following dimensionless formulas: $CV(x) = \dfrac{100 E_c h_0^3}{12 P_0 L^4} w_0(x)$, $GX(x) = \dfrac{100 E_c h_0^4}{12 P_0 L^4} \beta(x)$, where $h_0 = L/20$.

- The effect of the volume exponen *n:*

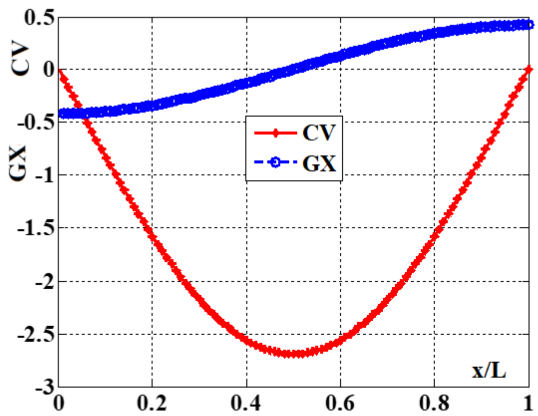
By varying the volume exponent *n* incrementally from 0 to 10, the calculated results for the beam's displacement and rotation are shown in Figure 2.

It is observed that as the volume exponent *n* increases, both the displacement and rotation of the beam also increase. This is because a larger volume exponent *n* corresponds to a higher proportion of metal in the beam, making the beam softer, and consequently, the displacement increases.

The displacement *w* reaches its maximum value at the mid-span of the beam, while the rotation angle achieves its minimum value at the two ends of the beam.



a. Maximum Displacement and Rotation

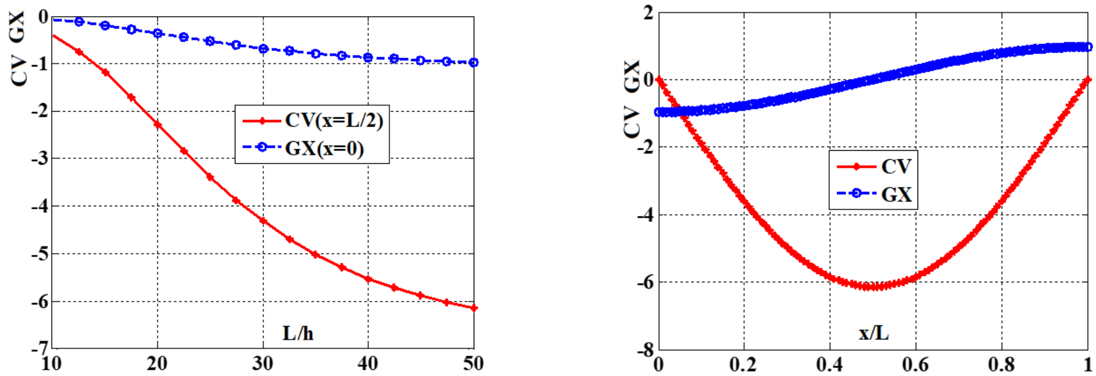


b. Displacement and Rotation Along the Entire Length of the Beam at n =10

Figure 2. Displacement and Rotation of the Beam as a Function of the Volume Exponent n, L/h=20, C_w=50, C_s=5.

Effect of Beam Thickness:

By varying the beam thickness h such that the ratio L/h increases from 10 to 50, the calculated results for the beam's displacement and rotation are presented in Figure 3. From this figure, it is observed that as the L/h ratio increases (the beam becomes thinner), the beam becomes more flexible, leading to increased displacement and rotation. The impact of thickness variation on the displacement w is more significant compared to its effect on the rotation.



a. Maximum Displacement and Rotation

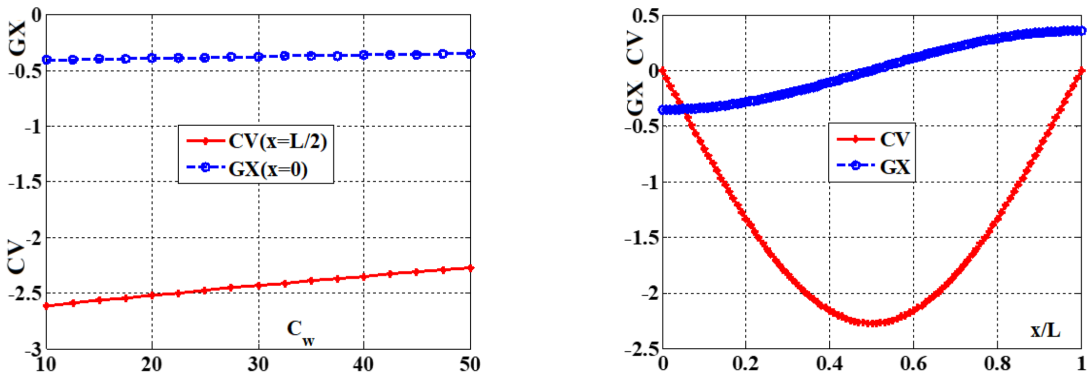
b. Displacement and rotation along the entire length of the beam at L/h =50

Figure 3. Displacement and Rotation of the Beam as a Function of the L/h, n=2, C_w=50, C_s=5.

Effect of Elastic Foundation Stiffness:

By varying the stiffness of the elastic foundation such that the coefficients C_w and C_s change from 10 to 50, the calculated results for the beam's displacement and rotation are presented in Figures 4 and 5. These results indicate that as the stiffness of the elastic foundation increases, the structure becomes stiffer, leading to reduced displacement and rotation of the beam.

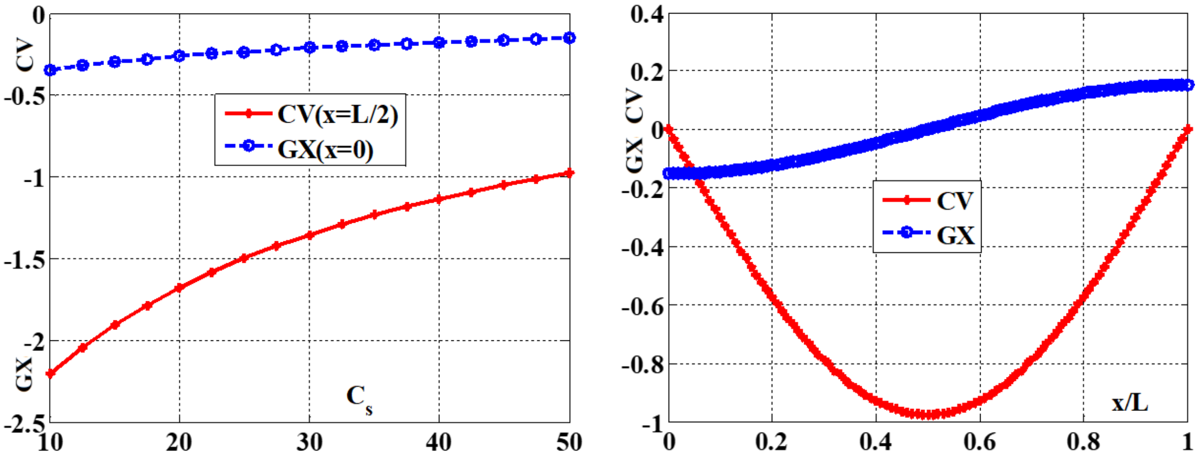
The influence of the stiffness parameter C_w on the displacement and rotation is less significant compared to the effect of C_s . This suggests that the displacement and rotation of the beam are more sensitive to the parameter C_s than to C_w.



a. Maximum Displacement and Rotation

b. Displacement and rotation along the entire length of the beam at C_w =50

Figure 4. Displacement and Rotation of the Beam as a Function of the Coefficient C_w, L/h =20, n=2, C_s=5.



a. Maximum Displacement and Rotation

b. Displacement and rotation along the entire length of the beam at C_s =50

Figure 5. Displacement and Rotation of the Beam as a Function of the Coefficient C_s, L/h =20, n=2, C_w=10.

5. CONCLUSION

Using the analytical method, the paper analyzed the displacement and rotation responses of functionally graded composite beams resting on an elastic foundation. Based on the calculated results, the paper draws the following key conclusions:

- The solution developed in this paper is reliable for solving the static bending problem of functionally graded composite beams.

- As the volume exponent n increases, the proportion of metal in the material also increases, leading to higher displacement and rotation of the beam.

- Smaller beam thickness and lower stiffness parameters of the elastic foundation result in greater displacement and rotation of the beam.

- The rotation of the composite beam reaches its maximum value at the two ends of the beam, whereas the displacement w reaches its maximum at the mid-span.

This is a novel study that opens up several potential research directions, such as dynamic analysis, random vibration, and optimization of geometry and material properties,….This study serves as a valuable reference for engineers in the design and fabrication of beams made from composite materials in practical applications.

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