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MONOMIAL ASYMPTOTIC POLYNOMIALS AND APPLICATIONS TO POLYNOMIAL OPTIMIZATION PROBLEMS

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Abstract. The problem of polynomial optimization plays an important role in many fields such as physics, chemistry, and economics. This problem has received research attention from many mathematicians recently. In this paper, we study the polynomial optimization problem over a non-compact semi-algebraic set, for which its constraint set of polynomials G is asymptotic with a finite family of monomials. By changing variables via a suitable monomial mapping, we transform the problem under consideration into the polynomial optimization problem over a compact semi-algebraic feasible set. We then apply the well-known result that the optimal value of a polynomial over a compact semi-algebraic set can be approximated as closely as desired by solving a hierarchy of semi-definite programs and the convergence is finite generically, to obtain results in the general case when the cone $C(G)$ is unimodular. In particular, in the case of polynomials in two variables, we solve the problem quite completely without requiring $C(G)$ to be unimodular.

Keywords: polynomial optimization, semi-algebraic, asymptotic, sum of squares

1. INTRODUCTION

We consider the problem of optimizing a polynomial function over a closed semi-algebra set that has the following form: Compute

$f_* = \inf_S f(x)$, with $S = [G \le s] := \{x \in \mathbb{R}^n | \{g_1(x) \le s_1, \dots, g_m(x) \le s_m\}$, where $G := \{g_1, \dots, g_m\}$ is a set of polynomials in $\mathbb{R}[x]$ and $s := \{s_1, \dots, s_m\}$ is a set of positive real numbers. Finding the global optimal value of a polynomial on a semi-algebraic is an NP-hard problem (see [1] or [2]). However, it is well-known that a systematic procedure has been established to solve polynomial optimization problems on compact basic semi-algebraic sets by Lasserre in [3]. It consists of a hierarchy of semidefinite programs of increasing size whose associated sequence of optimal values is monotone nondecreasing and converges to the global optimum. The proof of this convergence is based on powerful theorems from real algebraic geometry on the representation of polynomials that are positive on a basic semi-algebraic set via sum of squares, the so-called Positivstellensatze of Schmüdgen [4] and Putinar [5]. In addition, nice literature on the application of 'sums of squares and moment problem' to the polynomial optimization problems can be found in [6–8] and the references therein. However, when the feasible set is unbounded, the Schmüdgen and Putinar's Positivstellensatze do not hold anymore except in some special cases. Therefore, the convergence of the Lasserre's hierarchy cannot be guaranteed in general. A nice attempt to overcome the compact case is the work of Jeyakumar et al. [9], where a class of polynomial optimization problems with non-compact semi-algebraic feasible sets was studied. In their paper, the associated quadratic module, which is generated in terms of both the objective function and the constraints, is required to be Archimedean to show that the corresponding hierarchy converges and the convergence is generically finite.

We also consider in this paper a class of polynomial optimization problems on non-compact closed basic semi-algebraic feasible sets when their constraint polynomials are asymptotic with a family of monomials. The semi-algebraic sets under consideration are nice enough so that after changing variables via monomial mappings, these sets become compact. Furthermore, the optimal value of a polynomial over the original semi-algebraic set and that of the corresponding polynomial over the obtained compact semi-algebraic set are the same when some appropriate conditions are given. Therefore, the polynomial optimization problem on a non-compact semi-algebraic feasible set can be transformed into the one over the compact feasible set. This problem has been solved in the general case when the convex cone corresponding to the semi-algebraic set is unimodular as in [10]. In this paper, we solve the problem completely in the case of two variables without requiring that the convex cone corresponding to the semi-algebraic set is unimodular.

We organize our paper as follows: In Section 2, we give notations and the asymptotic concept of the polynomial family and survey some results on polynomial optimization problems, mainly in [10] and references therein. Section 3 is devoted to the study of the polynomial optimization problem on the set $[G \le s]$ when G is a family of polynomials in two variables, G is U-asymptotic and U generates the cone C(G), but without requiring C(G) to be unimodular as in [10].

2. PRELIMINARY

As usual, we denote the set of integers and the field of real numbers by $\mathbb{Z}$ and $\mathbb{R}$, respectively. Denote the ring of polynomials in $x = (x_1, \dots, x_n)$ with real coefficients by $\mathbb{R}[x]$. $\mathbb{Z}_+, \mathbb{N}, \mathbb{R}_+$ stand for the set of positive integers, the set of non-negative integers, and the set of positive real numbers, respectively. Let $G \coloneqq \{g_1, \dots, g_m\}$ be a set of polynomials in $\mathbb{R}[x]$ and $s := \{s_1, \dots, s_m\} \subset \mathbb{R}_+$. A semi-algebraic set corresponding to G is the set:

$$[\mathrm{G} \leq s] \coloneqq \{x \in \mathbb{R}^n | \{g_1(x) \leq s_1, \dots, g_m(x) \leq s_m\}. \quad (1)$$

Throughout the paper, $g_i(x)$ is always assumed to be zero for every i. Assume that the set $[G \leq s]$ is not empty. We consider the problem of minimizing a polynomial $f \in \mathbb{R}[x]$ on $[G \leq s]$. We are interested in computing the optimal value:

$$f_* \coloneqq inf\{f(x) | \, x \in [G \leq s]\}. \quad (2)$$

For $a = (a_1, a_2, \dots, a_n) \in \mathbb{N}^n$, we denote $x^a \coloneqq x_1^{a_1} \dots x_n^{a_n}$. In this paper, we are interested in the optimal problem when G is asymptotic with a set of monomials as the following definition.

Definition 1 (see [10, Definition 2.1]). Let $G \coloneqq \{g_1, \dots, g_m\}$ be a finite set of polynomials in $\mathbb{R}[x]$ and $U \subset \mathbb{N}^n$ a finite set of non-zero vectors. G is said to be U −asymptotic if there exist positive numbers c, M such that

$$c|x^a| \leq \max_{1 \leq i \leq m} g_i(x), \forall a \in U, x \in \mathbb{R}^n, \| x \| > M. \quad (3)$$

The inequality (3) is equivalent to

$$c \sum_{a \in U} |x^a| \leq \max_{1 \leq i \leq m} g_i(x), x \in \mathbb{R}^n, \| x \| > M. \quad (4)$$

We emphasize that constants c in (3) and (4) may be different. We also say that the G is asymptotic with the family of monomials $\{x^a | \, a \in U\}$.

We survey some results of the polynomial optimization problems on $[G \leq s]$ when the convex cone $C(U)$ is unimodular (as below definition) in [10], where $C(U)$ is the closed convex cone of the first orthant generated by U.

For $f \in \mathbb{R}[x]$, we can write $f(x) = \sum_{a \in \mathbb{N}^n} f_a x^a$, where $f_a \in \mathbb{R}$. The support of f is the set $supp(f) \coloneqq \{a \in \mathbb{N}^n | \, f_a \neq 0\}$. Let $G \coloneqq \{g_1, \dots, g_m\}$ be a subset of $\mathbb{R}[x]$. Then $supp(G)$ is defined as the union of the supports of all $g_i, i = 1,2, \dots, m$. $C(G)$ is denoted as the convex cone generated by $supp(G)$. Let $Y = \{y_1, \dots, y_k\}$ be a finite subset of $\mathbb{R}[x]$. We define the set

$$\mathbb{R}[Y] \coloneqq \{f(y_1, \dots, y_k) | \, f \in \mathbb{R}[x_1, \dots, x_k]\}.$$

For any $\mathbb{R} -$ subalgebra $\mathcal{A}$, we say that $\mathcal{A}$ is generated by $Y = \{y_1, \dots, y_k\}$ or Y generates $\mathcal{A}$ if $\mathcal{A} = \mathbb{R}[Y]$.

A convex cone C generated by p vectors $b^1, \dots, b^p$ in $\mathbb{N}^n$ is defined by:

$$C \coloneqq C(b^1, \dots, b^p) \coloneqq \{t_1 b^1 + \dots + t_p b^p | t_1 \geq 0, \dots, t_p \geq 0\}. \quad (5)$$

We can always assume that the greatest common divisor of the coordinates of each b^i is one for every $i = 1, \dots, p$. A convex cone $C \subset (\mathbb{R}_+)^n$ is said to be unimodular if there exists a generator set of n-vectors $a^1, \dots, a^n \in \mathbb{N}^n$ such that $det[a^1 \dots \, a^n] = 1$, where $[a^1 \dots \, a^n]$ is the matrix with a^i- its i^{th} column. We define the first parallelepiped of C to be the subset *H(C):*

$$H(C) \coloneqq \{t_1 b^1 + \cdots + t_p b^p | 0 \leq t_i \leq 1, \forall i = 1, \ldots, p\} \setminus \{b^1 + \cdots + b^p\}. \quad (6)$$

Lemma 1 (see [10], [11]). *Let $\mathcal{A}(C)$ denote the set of all polynomials with supports in the convex cone C. Then $\mathcal{A}(C)$ is an algebra generated by the finite set of many monomials $\{x^a | a \in H(C) \cap \mathbb{N}^n\}$, that is $\mathcal{A}(C) = \mathbb{R}[x^a | a \in H(C) \cap \mathbb{N}^n]$. Particularly, if C is unimodular generated by n-vectors $a^1, \ldots, a^n \in \mathbb{N}^n$ with $det[a^1 \ldots a^n] = 1$ then $\mathcal{A}(C) = \mathbb{R}[x^{a^1}, \ldots, x^{a^n}]$.*

Given a semi-algebraic set $S \subset \mathbb{R}^n$, denoted by $B(S)$ the set of all polynomials in $\mathbb{R}[x]$, which are bounded on S. Then, $B(S)$ is a real sub-algebra of $\mathbb{R}[x]$, and is sometimes called the bounded algebra for short.

Proposition 1 (see [10, Proposition 2.2]). *Let $G \coloneqq \{g_1, \ldots, g_m\}$ be a finite family of $\mathbb{R}[x]$, U be a finite subset of $C(G) \cap \mathbb{N}^n \setminus \{0\}$; $s \coloneqq (s_1, \ldots, s_m)$ be a sequence of m positive real numbers. If G is U-asymptotic and U generates $C(G)$ then*

$$B([G \leq s]) = \mathcal{A}(C(G)) = \mathcal{A}(C(U)).$$

Let $G \coloneqq \{g_1, \ldots, g_m\}$ be a finite family of $\mathbb{R}[x]$, by Lemma 1. Then we can assume that there exist k vectors $a^1, \ldots, a^k \in H(C(G)) \cap \mathbb{N}^n$ such that $\mathcal{A}(C(G)) = \mathbb{R}\left[x^{a^1}, \ldots, x^{a^k}\right]$. Put the matrix $A \coloneqq [a^1 \ldots a^k]$, where a^i is the i^{th}-column of A. We consider the mapping

$$\Phi(= \Phi_A): \mathbb{R}^n \longrightarrow \mathbb{R}^k, x \longmapsto u = x^A = (x^{a^1}, \ldots, x^{a^k}).$$

We call such mapping a monomial mapping. For any polynomial $f(x)$ in $\mathbb{R}[x]$, $f(x)$ belongs to $\mathcal{A}(C(G))$ if and only if there exists $\hat{f}(u)$ in $\mathbb{R}[u] = \mathbb{R}[u_1, \ldots, u_k]$ such that $f(x) = \hat{f}(x^A) = \hat{f}(\Phi(x))$. So, for each $i = 1, \ldots, m$, by $g_i \in \mathcal{A}(C(G))$, there exists $\hat{g}_i(\mathrm{u}) \in \mathbb{R}[\mathrm{u}]$ such that $g_i(x) = \hat{g}_i(x^A)$. We put the set

$$[\hat{G} \leq s] \coloneqq \{u \in \mathbb{R}^k | \hat{g}_i(u) \leq s_i, i = 1, \ldots, m\}.$$

In this paper, G is assumed to be U-asymptotic and $U \subset \mathbb{N}^n \setminus \{0\}$ generates $C(G)$. By Proposition 1, the bounded polynomials set $B([G \leq s])$ equals $\mathcal{A}(C(G)) = \mathbb{R}[x^A]$. Hence, for all $f(x)$ which is bounded on $[G \leq s]$, there also exists $\hat{f}(u)$ in $\mathbb{R}[u] = \mathbb{R}[u_1, \ldots, u_k]$ such that $f(x) = \hat{f}(x^A)$. Note that if $f(x)$ is in $B([G \leq s])$ then there exists its infimum on $[G \leq s]$.

Theorem 1 (see [10, Theorem 3.1]). *Let $S \coloneqq [G \leq s]$ be a basic closed semi-algebraic set defined as above. Assume that G is U-asymptotic, $C(U) = C(G)$, and the set $\hat{S} \coloneqq [\hat{G} \leq s]$ is the same as the closure $\overline{\hat{S} \cap (\mathbb{R} \setminus 0)^n}$ of $\hat{S} \cap (\mathbb{R} \setminus 0)^n$ in usual topology. Then*

(i). $\hat{S} = [\hat{G} \leq s]$ is a compact set,

(ii). For every polynomial f bounded on $S = [G \leq s]$, we have

$$\inf_S f(x) = \inf_{\hat{S}} \hat{f}(u) = \hat{f}(u^*) \text{ for some } u^* \in \hat{S}.$$

3. POLYNOMIALS IN TWO VARIABLES ASYMPTOTIC MONOMIALS AND APPLICATION

In this section, we consider the case of polynomials in two variables. Let $G \coloneqq \{g_1, \ldots, g_m\}$

be a finite subset of the ring of polynomials in two variables $\mathbb{R}[x] = \mathbb{R}[x_1, x_2]$, $s := (s_1, \dots, s_m)$ be a sequence of m positive real numbers. Then $[G \le s] := \{x \in \mathbb{R}^2 | g_1(x) \le s_1, \dots, g_m(x) \le s_m\}$ is a basic semi-algebraic set in $\mathbb{R}^2$. For the cone $C(G)$ in $\mathbb{R}^2$, there are two cases. The first case is that $C(G)$ is a one-dimensional cone. That means, there exists a vector $a \in \mathbb{N}^2 \setminus \{(0,0)\}$ such that $C(G) = C(a)$. The second case, $C(G)$ is a two-dimensional cone which means that there exist two linear independent vectors $a, b \in \mathbb{N}^2$ such that $C(G) = C(a, b)$. For each case, to solve the optimization problem on $[G \le s]$, we will follow the ideas in [10]. However, we have an improvement over [10], we solve the problem without requiring $C(G)$ to be unimodular.

3.1. The cone $C(G)$ is a one-dimensional cone

Assume that $C(G)$ is generated by the vector $a \in \mathbb{N}^2 \setminus \{(0,0)\}$, where the greatest common divisor of its coordinates is one. So, $\mathcal{A}\big(C(G)\big) = \mathbb{R}[x^a]$. For all $i = 1, \dots, m$, then $g_i(x) = \hat{g}_i(x^a)$, where $\hat{g}_i$ is a single variable polynomial. The set $\big[\hat{G} \le s\big] := \{t \in \mathbb{R} | \hat{g}_i(t) \le s_i, i = 1, \dots, m\}$ is a subset of $\mathbb{R}$.

Proposition 2. *Assume that $C(G) = C(a)$ is as above. Let k be a positive integer. Then the following statements are equivalent:*

(i). *G is ka −asymptotic. This means that there exist two positive real numbers c, M such that*

$$c|x^{ka}| \le \max_{1 \le i \le m} g_i(x), \forall x \in \mathbb{R}^2, \| x \| \ge M.$$

(ii). *There exists a positive real number c such that the set $\hat{G} = \{\hat{g}_1, \dots, \hat{g}_m\}$ satisfies*

$$c|t^k| \le \max_{1 \le i \le m} \hat{g}_i(t), \forall t \in \mathbb{R}.$$

Proof:

It is straightforward that (i) follows from (ii).

Assume that (i) holds, then the following 3 cases occur:

Case 1: $a = (1,0)$. For every $t \in \mathbb{R}$, we put $x = (t, M)$. Then $\| x \| \ge M$ and $t = x^a$. By the definition of $\hat{g}_i$ and the assumption (i), we get:

$$\max_{1 \le i \le m} \hat{g}_i(t) = \max_{1 \le i \le m} g_i(x) \ge c|x^{ka}| = c|t^k|.$$

Case 2: $a = (0,1)$. The proof of this case is similar to Case 1.

Case 3: $a = (a_1, a_2) \in (\mathbb{N} \setminus \{0\})^2$, where the greatest common divisor of a_1 and a_2 is one. We can assume that a_1 is an odd number. If $x = (t^{\frac{1}{a_1}} M^{\frac{-a_2}{a_1}}, M)$, then $x^a = t$ and $\| x \| \ge M$. From the assumption (i), we get (ii). □

Theorem 2. *Let $S := [G \le s]$ be a basic closed semi-algebraic set in $\mathbb{R}^2$ defined as above, k be a positive integer and $a \in \mathbb{N}^2 \setminus \{(0,0)\}$. Assume that G is ka-asymptotic, $C(a) = C(G)$, and the set $\hat{S} := \big[\hat{G} \le s\big]$ is defined as above. Then we have the following statements:*

(i). *$\hat{S} = \big[\hat{G} \le s\big]$ is a compact set,*

(ii). *For every polynomial f bounded on $S = [G \le s]$, there exists a single variable polynomial $\hat{f}$ such that $f(x) = \hat{f}(x^a)$ and*

$$\inf_{S} f(x) = \inf_{\hat{S}} \hat{f}(t) = \hat{f}(t_*) \text{ for some } t_* \in \hat{S}.$$

To prove the above theorem, we use the following claim.

Claim 1. *For a in $\mathbb{N}^2 \setminus \{(0,0)\}$ as in Theorem 2, consider the mapping $\Phi: \mathbb{R}^2 \longrightarrow \mathbb{R}; x \mapsto x^a$. Then $\Phi([G \leq s]) = [\hat{G} \leq s]$.*

Indeed, by $g(x) = \hat{g}(x^a) = \hat{g}(\Phi(x))$, obviously $\Phi([G \leq s]) \subset [\hat{G} \leq s]$. If $a = (1,0)$, for every $t \in [\hat{G} \leq s]$, we put $x = (t, 1)$. Then $t = x^a = \Phi(x)$ and $x \in [G \leq s]$. So $t \in \Phi([G \leq s])$. If $a = (0,1)$, for every $t \in [\hat{G} \leq s]$, we put $x = (1, t)$. Then $x \in [G \leq s]$ and $t = x^a = \Phi(x) \in \Phi([G \leq s])$. If $a = (a_1, a_2) \in (\mathbb{N} \setminus \{0\})^2$, we also assume that a_1 is an odd number. For every $t \in [\hat{G} \leq s]$, we take $x = (t^{\frac{1}{a_1}}, 1)$, then $t = x^a = \Phi(x) \in \Phi([G \leq s])$. Hence $[\hat{G} \leq s] \subset \Phi([G \leq s]) t \in [\hat{G} \leq s]$.

Proof of Theorem 2:

(i) Since G is ka-asymptotic and $\mathrm{C}(a) = \mathrm{C}(\mathrm{G})$, applying Proposition 1, we see that the set $\mathrm{B}([\mathrm{G} \leq \mathrm{s}])$ of polynomials bounded on $[\mathrm{G} \leq \mathrm{s}]$ is equal to the set $\mathcal{A}(\mathrm{C}(\mathrm{G}))$ of polynomials supported on $C(G)$. By $a \in \mathrm{C(a)} = \mathrm{C(G)}$, the polynomial x^a is bounded on $[\mathrm{G} \leq \mathrm{s}]$. It means that there exists $L > 0$ such that $|x^a| \leq L, \forall x \in [G \leq s]$. For every $\mathrm{t} \in [\hat{\mathrm{G}} \leq \mathrm{s}]$, by Claim 1, there exists $\mathrm{y} \in [\mathrm{G} \leq \mathrm{s}]$ such that $\mathrm{t} = \mathrm{y}^{\mathrm{a}}$. So $|t| = |y^a| \leq L, \forall t \in [\hat{\mathrm{G}} \leq \mathrm{s}]$. Therefore, $[\hat{\mathrm{G}} \leq \mathrm{s}]$ is compact.

(ii) According to Proposition 1, $\mathrm{B}([\mathrm{G} \leq \mathrm{s}]) = \mathcal{A}(\mathrm{C}(\mathrm{G}))$. Thus, for f bounded on $\mathrm{S} = [\mathrm{G} \leq \mathrm{s}]$, there exists a single variable polynomial $\hat{f}$ such that $f(x) = \hat{f}(x^a)$. Moreover, by Claim 1 and *(i)*, we get

$$\inf_{x \in S} f(x) = \inf_{x \in S} \hat{f}(\Phi(x)) = \inf_{t \in \Phi(S)} \hat{f}(\mathrm{t}) = \inf_{t \in \hat{S}} \hat{f}(\mathrm{t}) = \hat{f}(t_*) \text{ for some } t_* \in \hat{S}. \qquad \square$$

Example 1. Find the infimum of the polynomial $f(x_1, x_2) = 2x_1^5 x_2^5 + 5x_1^2 x_2^2$ on the set

$$S = \{(x_1, x_2) \in \mathbb{R}^2 |\ x_1^4 x_2^4 + x_1^2 x_2^2 \leq 2\}.$$

We have $S = [g \leq 2]$, where $g(x_1, x_2) = x_1^4 x_2^4 + x_1^2 x_2^2$ is 2(1,1)-asymptotic and $C((1,1)) = C(g)$. The polynomials $\hat{f}(t) = 2t^5 + 5t^2; \hat{g}(t) = t^4 + t^2$ satisfy $f(x_1, x_2) = \hat{f}(x_1 x_2)$; $g(x_1, x_2) = \hat{g}(x_1 x_2)$. And the set

$$\hat{S} = [\hat{g} \leq 2] = \{t \in \mathbb{R} |\ t^4 + t^2 \leq 2\} = \{t \in \mathbb{R} | -1 \leq t \leq 1\},$$

is a compact set in $\mathbb{R}$. By Theorem 2, we get

$$\inf_{S} f = \inf_{\hat{S}} \hat{f} = 0 = \hat{f}(0).$$

3.2. The cone $C(G)$ is a two-dimensional cone

In this subsection, we assume that $C(G)$ is a two-dimensional cone which means that there exist two linear independent vectors $a, b \in \mathbb{N}^2 \setminus \{(0,0)\}$ such that $C(G) = C(a, b)$. Note that both of the greatest common divisors of the corresponding coordinates of a and b are equal to one. If $\det[a\ b] = 1$ then $C(a, b)$ is unimodular and the problem of finding the infimum of a polynomial on $[G \leq s]$ is carried out as in [10]. Now, we assume $\det[a\ b] = p > 1$. Then $H(C(a, b)) = \{t_1 a + t_2 b |\ 0 \leq t_1, t_2 \leq 1\} \setminus \{a + b\}$ has p integer points, except two points a

and b (An integer point is a point whose coordinate parts are integer). We have the following lemma.

Lemma 2. *If $det[a\ b] > 1$ then there exists $c \in H(C(a,b)) \cap \mathbb{N}^2$ such that $det[a\ c] = 1$ and $C(a,c)$ does not contain any integer points of $H(C(a,b))$, except three points a, c and the origin 0.*

Proof.

Using the property of a convex set and $det[a\ b] > 1$, we can take an integer point $c \in H(C(a,b)) \cap \mathbb{N}^n$ such that the convex cone $C(a,c)$ does not contain any integer points of $H(C(a,b))$. So, the convex hull $\Delta(a,c) \subset H(C(a,b))$ of $\{0, a, c\}$ (the smallest convex set containing $\{0, a, c\}$) does not have any integer points, except three points a, c and the origin 0. We denote that convex hull by $\Delta(a,c)$, then $\Delta(a,c) = \{t_1 a + t_2 c |\ t_1, t_2 \geq 0, t_1 + t_2 \leq 1\}$. If there is an integer point $e = t_1 a + t_2 c \in H(C(a,c)) \backslash \{0, a, c\}$, where $0 < t_1, t_2 < 1$, then the integer point $a + c - e = (1 - t_1)a + (1 - t_2)c$ belongs to $\Delta(a,c) \backslash \{0, a, c\}$. Hence, $H(C(a,c)) \backslash \{0, a, c\}$ does not contain any integer points. That is, $\det[a\ c] = 1$. □

Applying Lemma 1 and Lemma 2, there exists a finite set of distinct non-zero vector $a^1, \dots, a^k$ in $H(C(a,b)) \cap \mathbb{N}^2$ with $a^1 = a,\ a^2 = c,\ a^3 = b,\ \det[a^1\ \ a^2] = 1, a^i \notin C(a^1, a^2)\ \forall i \geq 3$, such that:

$$\mathcal{A}(C(G)) = \mathcal{A}(C(a,b)) = \mathbb{R}\left[x^{a^1}, \dots, x^{a^k}\right].$$

Put the matrix $A = [a^1\ \dots\ a^k]$ -a matrix of size $2 \times k$, where every vector a^i is a column of A. We denote $(x^{a^1}, \dots, x^{a^k})$ by x^A. We consider the mapping

$$\Phi (or\ \Phi_{\mathrm{A}}): \mathbb{R}^2 \to \mathbb{R}^k, x \mapsto u = \Phi(x) = x^A.$$

A polynomial $f \in \mathbb{R}[x] = \mathbb{R}[x_1, x_2]$ belongs to $\mathbb{R}[x^A] = \mathcal{A}(C(G))$ if and only if there exists $\hat{f} \in \mathbb{R}[u_1, u_2, \dots, u_k]$ such that $f(x) = \hat{f}(\Phi(x)) = \hat{f}(x^A)$. So, for every $g_i, i = 1,2, \dots, m$, there exists $\hat{g}_i \in \mathbb{R}[u]$ such that $g_i(x) = \hat{g}_i(x^A)$. We put $[\hat{G} \leq s] = \{u \in \mathbb{R}^k |\ \hat{g}_i(u) \leq s_i, \forall i = 1,2, \dots, m\}$.

Since $a^1, a^2, a^3, \dots, a^k \in H(C(a,b)) \cap \mathbb{N}^2, a^1 = a, \det[a^1\ a^2] = 1$ and $a^i \notin C(a^1, a^2)\ \forall i \geq 3$, there exist positive integers q_{i1}, q_{i2} for $i = 3, 4, \dots, k$ such that

$$a^i = -q_{i1} a^1 + q_{i2} a^2, \qquad i = 3, 4, \dots, k. \tag{7}$$

And, since $a^2, a^4, a^5, \dots, a^k \in H(C(a,b)) \cap \mathbb{N}^2 = H(C(a^1, a^3)) \cap \mathbb{N}^2$, there exist positive integers p_j, p_{j1}, p_{j2} for $j = 2,4,5, \dots, k$ such that

$$p_j a^j = p_{j1} a + p_{j2} b = p_{j1} a^1 + p_{j2} a^3, \qquad j = 2,4,5, \dots, k. \tag{8}$$

Note that we can choose p_j, p_{j1}, p_{j3} such that the greatest common divisor of every set $\{p_j, p_{j1}, p_{j3}\}$ equals one.

For all $x \in \mathbb{R}^2$, by (7) and (8), if $u = x^A \Leftrightarrow u_i = x^{a^i}$, then u satisfies the following equations:

$$u_i u_1^{q_{i1}} = u_2^{q_{i2}}, \qquad i = 3, 4, \dots, k; \tag{9}$$

$$u_j^{p_j} = u_1^{p_{j1}} u_3^{p_{j2}}, \quad j = 2,4,5, \dots, k. \tag{10}$$

For positive integers $q_{i1}, q_{i2};\ p_j, p_{j1}, p_{j3}$ above, we put

$$\Omega := \left\{u \in \mathbb{R}^k \,\middle|\, u_i u_1^{q_{i1}} = u_2^{q_{i2}}, i = 3, \dots, k;\ u_j^{p_j} = u_1^{p_{j1}} u_3^{p_{j2}}, j = 2,4,5, \dots, k\right\}. \quad (11)$$

That is, Ω is a semi-algebraic set and is the solution set of the system of equations (9) and (10).

Lemma 3. *Let the set $[G \leq s]$ and the mapping Φ be defined as above. Then:*

(i) $\Phi(\mathbb{R}^2) \subset \Omega;\ \Phi((\mathbb{R}^*)^2) = \Omega \cap (\mathbb{R}^*)^k$, *where* $\mathbb{R}^* = \mathbb{R} \setminus \{0\}$

(ii) $\overline{\Phi(\mathbb{R}^2)} = \Omega$.

Proof:

(i) By the equations (7) and (8), if $x \in \mathbb{R}^2$ then $u = \Phi(x) = x^A$ satisfy the equations (9) and (10), so $x \in \Omega$.

Obviously, $\Phi((\mathbb{R}^*)^2) \subset \Omega \cap (\mathbb{R}^*)^k$. For the reverse inclusion, take $u = (u_1, \dots, u_k) \in \Omega \cap (\mathbb{R}^*)^k$ and put the square matrix $B = [a^1\ a^2]$. By $\det B = 1$, there exists the inverse matrix B^{-1} of integer entries. Put $x = (x_1, x_2) = (u_1, u_2)^{B^{-1}} \in (\mathbb{R}^*)^2$, then $(u_1, u_2) = x^B = (x^{a^1}, x^{a^2})$. From the equalities (7) and (9), we have

$$u_i = u_1^{-q_{i1}} u_2^{q_{i2}} = x^{-q_{i1}a^1 + q_{i2}a^2} = x^{a^i},\ i = 3, 4, \dots, k.$$

So $u = x^A = \Phi(x) \in \Phi((\mathbb{R}^*)^2)$. Hence, $\Phi((\mathbb{R}^*)^2) = \Omega \cap (\mathbb{R}^*)^{\mathrm{k}}$.

(ii) Clearly, $\overline{\Phi(\mathbb{R}^2)} \subset \Omega$. Conversely, to prove that $\Omega \subset \overline{\Phi(\mathbb{R}^2)}$, by (i), we only need to show that if $u \in \Omega \setminus (\mathbb{R}^*)^k$ then $u \in \overline{\Phi(\mathbb{R}^2)}$. The following cases occur:

Case 1: $u = (0,0, \dots, 0) \in \Omega \setminus (\mathbb{R}^*)^k$. We have $u = (0,0)^A = \Phi(0,0)$. So $u \in \Phi(\mathbb{R}^2)$.

Case 2: $u = (u_1, 0,0, \dots, 0) \in \Omega \setminus (\mathbb{R}^*)^k; u_1 \neq 0$. For every real number $t \neq 0$, we can determine $u(t) = \big(u_1(t), u_2(t), \dots, u_k(t)\big) \in (\mathbb{R}^*)^k$, where $u_1(t) = u_1, u_2(t) = t,\ u_i(t) = u_1^{-q_{i1}} t^{q_{i2}}$ for all $i = 3,4, \dots, k$. We see that $u(t)$ satisfy the equations in (9). In a similar way as in the proof (i), it follows that there exists $x(t) = \big(u_1(t), u_2(t)\big)^{B^{-1}} \in (\mathbb{R}^*)^2$, then $\Phi\big(x(t)\big) = u(t)$. So, $u(t)$ is a solution of the system of equations (10) for all $t \neq 0$. Therefore, $u(t) \in \Omega \cap (\mathbb{R}^*)^k$. Moreover, $\Phi\big(x(t)\big) = u(t) \to u$ as $t \to 0$. So, $u = (u_1, 0,0, \dots, 0) \in \overline{\Phi(\mathbb{R}^2)}$.

Case 3: $u = (0,0, u_3, 0, \dots, 0) \in \Omega \setminus (\mathbb{R}^*)^k; u_3 \neq 0$. Since $a_3 = -q_{31}a^1 + q_{32}a^2$ in (7), if q_{32} is an even number then q_{31} is an odd number because the corresponding coordinate components of each a^1, a^2 and a^3 all have at least one odd number (by the greatest common divisor of the coordinate components of each a^i equals 1). So, for every $t \neq 0$, we can put $\overline{u_1}(t) = u_3 t^2$ and $\overline{u_2}(t) = (u_3^{q_{31}+1} t^{2q_{31}})^{1/q_{32}}$. By $\overline{u_1}(t) \neq 0, \overline{u_2}(t) \neq 0$, we can determine $x(t) = \big(\overline{u_1}(t), \overline{u_2}(t)\big)^{B^{-1}}$. Then $x(t)$ is in $(\mathbb{R}^*)^2$ and $\overline{u_1}(t) = x(t)^{a^1}, \overline{u_2}(t) = x(t)^{a^2}$. Putting $u(t) = \Phi\big(x(t)\big)$, it is obvious that $u(t) \in \Omega$. According to the definitions of Φ and $x(t)$, we get $u_1(t) = x(t)^{a^1} = u_3 t^2, u_2(t) = x(t)^{a^2} = (u_3^{q_{31}+1} t^{2q_{31}})^{1/q_{32}}, u_i(t) = x(t)^{a^i}\ \forall i = 1, \dots, k$. We show that $u(t) \to (0,0, u_3, 0, \dots, 0)$ as $t \to 0$. Indeed, clearly $u_1(t) \to 0, u_2(t) \to 0$. Since the coordinates of $u(t)$ satisfy the equations (9), we have $u_3(t) = u_1(t)^{-q_{31}} u_2(t)^{q_{32}} = u_3$. Since the coordinates of $u(t)$ satisfy the equations (10), we obtain

$u_j(t)^{p_j} = u_1(t)^{p_{j1}} u_3(t)^{p_{j2}} = u_3^{p_{j1}+p_{j2}} t^{2p_{j1}} \to 0$ for all $j = 3,4,\dots,k$. So $u_j(t) \to 0$ as $t \to 0$ for all $j = 3,4,\dots,k$. Therefore, $\Phi\big(x(t)\big) = u(t) \to (0,0,u_3,0,\dots,0)$.

Case 4: $0 \neq u \in \Omega \setminus (\mathbb{R}^*)^k$. Then there exists $u_i = 0$ for some i. If $i = 1$ $(i.e., u_1 = 0)$, then $u_j = 0\ \forall j \neq 1,3$ (by Equations (10)) and $u_3 \neq 0$. By Case 3, $u \in \overline{\Phi(\mathbb{R}^2)}$. If $i = 3$ $(i.e., u_3 = 0)$, then $u_j = 0, \forall j \neq 1,3$ (by Equation (10)) and $u_1 \neq 0$. By Case 1, $u \in \overline{\Phi(\mathbb{R}^2)}$. If $i \neq 1$ and $i \neq 3$ $(i.e. \exists\, i \neq 1, i \neq 3,\ u_i = 0)$, then $u_1 = 0$ or $u_3 = 0$ (by Equation (10)). According to the Case 2 or Case 3, u belongs to $\overline{\Phi(\mathbb{R}^2)}$.

Theorem 3. *Let $S := [G \leq s]$ be a basic closed semi-algebraic set in $\mathbb{R}^2, C(G) = C(a,b)$, the matrix A, the mapping Φ, the set Ω, the set $\hat{S} := \big[\hat{G} \leq s\big]$ all defined as in this subsection. Let k_1, k_2 be positive integers. Assume that G is $\{k_1 a, k_2 b\}$-asymptotic and* 0 *is not a local minimal value of the function $h(u) = \max\limits_{1\leq i\leq m} (\hat{g}_i(u) - s_i)$ on Ω. Then*

(i) $\hat{S} \cap \Omega = \overline{\Phi(S)}$, *where $\overline{\Phi(S)}$ is the closure of the set $\Phi(S)$.*

(ii) $\hat{S} \cap \Omega$ *is a compact set.*

(iii) *For every polynomial f bounded on $S = [G \leq s]$, there exists a polynomial $\hat{f}$ such that $f(x) = \hat{f}(x^A)$ and*

$$\inf_S f(x) = \inf_{\hat{S}\cap\Omega} \hat{f}(t) = \hat{f}(u_*)\ for\ some\ u_* \in \hat{S} \cap \Omega.$$

Proof.

(i) We always have $\overline{\Phi(S)} \subset \hat{S} \cap \Omega$.

Now, we prove that $\hat{S} \cap \Omega$ is contained in $\overline{\Phi(S)}$. Take arbitrarily $u^0 \in \hat{S} \cap \Omega$. By $u^0 \in \Omega$ and Lemma 3, there exists a sequence $\{u^n\}_{n\in\mathbb{N}} \subset \Phi(\mathbb{R}^2)$ such that $u^n \to u^0$ as $n \to \infty$. That is, there exists $\{x^n\}_{n\in\mathbb{N}} \subset \mathbb{R}^2$ such that $u^n = \Phi(x^n) \to u^0$ as $n \to \infty$. By $u^0 \in \hat{S}$, we have $h(u^0) \leq 0$. We consider 2 cases:

Case 1: $h(u^0) < 0$. Since h is continuous, there exists an open ball $\mathcal{B}(u^0,\epsilon)$ centered at u^0 with radius $\epsilon > 0$ such that $h(u) < 0$ for all $u \in \mathcal{B}(u^0,\epsilon)$. Since $u^n \to u^0$, there exists an integer number $0 < N \in \mathbb{N}$ such that $u^n \in \mathcal{B}(u^0,\epsilon)$ for all $n \geq N$. So $h(u^n) < 0$. That means, $\hat{g}_i(u^n) < s_i$ for all $i = 1,2,\dots,m$ and $n \geq N$. Furthermore, for all $n \geq N$, we have:

$$g_i(x^n) = \hat{g}_i\big(\Phi(x^n)\big) = \hat{g}_i(u^n) \leq s_i\ \forall i = 1,2,\dots,m.$$

Hence, $u^n \in \Phi(S)\ \forall n \geq N$ and $u_n \to u^0$ as $n \to \infty$. So $u^0 \in \overline{\Phi(S)}$.

Case 2: $h(u^0) = 0$. Assume that $u^0 \notin \overline{\Phi(S)}$, there exists an open ball $\mathcal{B}(u^0,\epsilon)$ centered at u^0 with radius $\epsilon > 0$ such that $\mathcal{B}(u^0,\epsilon) \cap \overline{\Phi(S)} = \emptyset$. By Case 1, $h(u) \geq 0 = h(u^0)$ for all $u \in \mathcal{B}(u^0,\epsilon) \cap \Omega$. That is, 0 is a local minimal value of h on Ω, which contradicts the hypothesis. So, $u^0 \in \overline{\Phi(S)}$.

(ii) By G is $\{k_1 a, k_2 b\}$-asymptotic, $C(k_1 a, k_2 b) = C(a,b) = C(G)$ and Proposition 1, $B([G \leq s]) = \mathcal{A}\big(C(G)\big) = \mathbb{R}[x^A] = \mathbb{R}\left[x^{a^1},\dots,x^{a^k}\right]$. So, every monomial x^{a^i} is bounded on $[G \leq s]$ for all $i = 1,2,\dots,k$. There exists $M_i > 0$ such that $\left|x^{a^i}\right| \leq M_i$ for all $x \in S = [G \leq s]$

and $i = 1, 2, \dots, k$. Take arbitrarily $u = (u_1, \dots, u_k) \in \hat{S} \cap \Omega$. By $\hat{S} \cap \Omega = \overline{\Phi(S)}$ (follows (i)), there exists a sequence $\{x^n\} \subset S$ such that $\Phi(x^n) \to u$ as $n \to \infty$, that means $(x^n)^{a^i} \to u_i, \forall i = 1,2, \dots, k$. According to the above argument, $\left|(x^n)^{a^i}\right| \leq M_i, \forall i = 1,2, \dots, k$. So $|u_i| \leq M_i, \forall i = 1,2, \dots, k$. Hence, $\hat{S} \cap \Omega$ is compact.

(iii) By $f \in B([G \leq s])$ and $B([G \leq s]) = \mathcal{A}(C(G))$ (apply Proposition 1), there exists $\hat{f} \in \mathbb{R}[u]$ such that $f(x) = \hat{f}(x^A)$. Since *(i)* and *(ii)* we have

$$f_* = \inf_S f(x) = \inf_{\overline{\Phi(S)}} \hat{f}(u) = \inf_{\hat{S} \cap \Omega} \hat{f}(u) = \hat{f}(u_*) \text{ for some } u_* \in \hat{S} \cap \Omega. \qquad \square$$

Note that, in Theorem 3 , $\hat{f}$ attains its minimum value on $\hat{S} \cap \Omega$, but f may not attain its minimum value on S.

Example 2. Find the infimum of $f(x) = f(x_1, x_2) = x_1^4 x_2^4 - x_1^4 x_2^8$ on the set $S = [G \leq 1] = \{x_1^2 \leq 1, x_1^4 x_2^8 \leq 1\}$.

We have that $G = \{x_1^2, x_1^4 x_2^8\}$ is {2(1,0), 4(1,2)}-asymptotic, $C(G)$ equals $C\big((1,0), (1,2)\big)$. So $B([G \leq s])$ is the same $\mathcal{A}(C(G)) = \mathbb{R}[x_1, x_1 x_2, x_1 x_2^2]$. Put the matrix $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix}$, the mapping $\Phi: \mathbb{R}^2 \to \mathbb{R}^3$, $\Phi(x) = x^A = (x_1, x_1 x_2, x_1 x_2^2)$. Put $\hat{f}(u) = \hat{f}(u_1, u_2, u_3) = u_2^4 - u_3^4$ then $f(x) = \hat{f}(x^A)$. Put $\big[\hat{G} \leq 1\big] \cap \Omega = \{u \in \mathbb{R}^3 |\, u_1^2 \leq 1, u_3^4 \leq 1, u_2^2 = u_1 u_3\} = \{-1 \leq u_1 \leq 1, -1 \leq u_3 \leq 1,\ u_2^2 = u_1 u_3\}$, then $\big[\hat{G} \leq 1\big] \cap \Omega$ is compact in $\mathbb{R}^3$. It is easy to check that $\overline{\Phi([G \leq 1])} = \big[\hat{G} \leq 1\big] \cap \Omega$. By Theorem 3, we get

$$\inf_{[G \leq 1]} f = \min_{[\hat{G} \leq 1] \cap \Omega} \hat{f} = -1 = \hat{f}(0,0,1).$$

However, f does not attain its minimum on $[G \leq 1]$.

4. CONCLUSION

In this paper, we have studied the problem of minimizing a polynomial $f \in \mathbb{R}[x]$ on non-compact semi-algebraic set $[G \leq s]$ when the family of polynomials G is asymptotic with a family of monomials $\{x^a |\, a \in U\}$ and $U \subset \mathbb{N}^n$ generates the convex cone $C(G)$. At this time, the algebra $B([G \leq s])$ of polynomials bounded on $[G \leq s]$ equals the algebra $\mathcal{A}(C(G))$ of all polynomials with supports in the convex cone C(G). Under certain suitable conditions and by making changes of variables, the problem becomes the problem of minimizing the polynomial $\hat{f}$ on compact semi-algebraic set $\big[\hat{G} \leq s\big]$. In the general case of n variables, the problem was solved provided that $C(G)$ is an unimodular cone. In the case of two variables, we have solved the problem without requiring $C(G)$ to be unimodular.

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