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UPPER BOUND ON THE NUMBER OF DETERMINING MODES FOR THE 2D g-BÉNARD PROBLEM

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Abstract. The "determining modes" concept introduced by Foias and Prodi in 1967 say that if two solutions agree asymptotically in their P projection, then they are asymptotical in their entirety. In this paper, we consider the 2D g-Bénard problem in domains satisfying the Poincaré inequality with homogeneous Dirichlet boundary conditions. We present an improved upper bound on the number of determining modes. Moreover, we slightly improve the estimate on the number of determining modes and obtain an upper bound of the order G. These estimates are in agreement with the heuristic estimates based on physical arguments, that have been conjectured by O.P. Manley and Y.M. Treve. The Gronwall lemma and Poincaré type inequality will play a central role in our computational technique as well as the proof of the main result of the paper. Studying the properties of solutions is important to determine the behavior of solutions over a long period of time. The obtained results particularly extend previous results for 2D g-Navier-Stokes equations and 2D Bénard problem.

Keywords: g-Bénard problem, determining modes, Grashof number.

1. INTRODUCTION

Let Ω be a (not necessarily bounded) domain in $\mathbb{R}^2$ with boundary Γ. We consider the following two-dimensional (2D) g -Bénard problem

$$\begin{cases} \dfrac{\partial u}{\partial t} + (u\cdot\nabla)u - \nu\Delta u + \nabla p = \xi\theta + f,\ x\in\Omega,\ t>0, \\ \nabla\cdot(gu) = 0,\ x\in\Omega,\quad t>0, \\ \dfrac{\partial\theta}{\partial t} + (u\cdot\nabla)\theta - \kappa\Delta\theta - \dfrac{2\kappa}{g}(\nabla g\cdot\nabla)\theta - \dfrac{\kappa\Delta g}{g}\theta = \tilde{f},\ x\in\Omega,\ t>0, \\ u = 0,\ x\in\Gamma,\quad t>0, \\ \theta = 0,\ x\in\Gamma,\quad t>0, \\ u(x,0) = u_0(x),\quad x\in\Omega, \\ \theta(x,0) = \theta_0(x),\quad x\in\Omega, \end{cases} \tag{1}$$

where $u\equiv u(x,t)=(u_1,u_2)$ is the unknown velocity vector, $\theta\equiv\theta(x,t)$ is the temperature, $p\equiv p(x,t)$ is the unknown pressure, f is the external force function, $\tilde{f}$ is the heat source function, $\nu>0$ is the kinematic viscosity coefficient, ξ is a constant vector, $\kappa>0$ is thermal diffusivity, u_0 is the initial velocity and θ_0 is the initial temperature.

The g-Bénard problem is a variation of the Bossiness equations which consists of a system that couples Navier-Stokes and advection-diffusion heat in order to model convection in a fluid [1-12]. Moreover, when $g\equiv\mathrm{const}$ we get the usual Bénard problem, and when $\theta\equiv 0$ we get the g-Navier-Stokes equations. The 2D g-Bénard problem arises when we study the usual 3D Boussinesq equations on thin domains $\Omega_g=\Omega\times(0,g)$. In what follows, we list some related results.

The conventional theory of turbulence asserts that turbulent flows are monitored by a finite number of degrees of freedom [1-12]. The notion of determining modes in [2,4,5] is rigorous attempts to identify those parameters that control turbulent flows. The theory of determining modes was introduced by Foias and Prodi in 1967 [4]. Jones and Titi presented improved upper bounds on the number of determining Fourier modes, determining nodes and volume elements for the Navier - Stokes equations in [6]. The dependence of the number of numerically determining modes in the Navier - Stokes equations on the Grashof number is examined in [7,8]. Then, in [3,9] M.Özlük and M. Kaya also study the number of determining modes to 2D g-Bénard problem for periodic time boundary conditions.

Studying the properties of solutions is important to determine the behavior of solutions over a long period of time.

We will study the number of determining modes to 2D g-Bénard problem in domains that are not necessarily bounded but satisfy the Poincaré inequality. In particular, the dependence of the number of deterministic modes of numbers on Grashof's numbers. To do this, we assume that the domain Ω and functions $f,\tilde{f},g$ satisfy the following hypotheses:

(Ω) Ω is an arbitrary (not necessarily bounded) domain in $\mathbb{R}^2$ satisfying the Poincaré type inequality

$$\int_{\Omega} \phi^2 g dx \le \frac{1}{\lambda_1} \int_{\Omega} |\nabla \phi|^2 \, g dx, \quad \text{for all } \phi \in C_0^\infty(\Omega); \tag{2}$$

where $\lambda_1 > 0$ is the first eigenvalue of the g-Stokes operator in Ω;

(F) $f \in L^2(0,T;H_g)$, $\tilde{f} \in L^2(0,T;L^2(\Omega,g))$;

(G) $g \in W^{1,\infty}(\Omega)$ such that

$$0 < m_0 \le g(x) \le M_0 \text{ for all } x = (x_1, x_2) \in \Omega, \text{ and } |\nabla g|_\infty^2 < m_0^2 \lambda_1, \tag{3}$$

where $\lambda_1 > 0$ is the constant in the inequality (1.2).

The article is organized as follows. In Section 2, for convenience of the reader, we recall the functional setting of the 2D g-Bénard problem. Section 3 we show the number of determining modes.

2. PRELIMINARIES

In this paper, we use the following function spaces:

- $C_0^\infty(\Omega_g)$ is a space of all infinitely differentiable functions with compact support in Ω_g.
- $L^P(\Omega_g) = \{u : \Omega_g \to \mathbb{R}^n | u \ \tau \in W_g \mapsto (\nabla\theta, \nabla\tau)_g \in$ is a Lebesgue measured function and $1 \le p < +\infty$; $\int_{\Omega_g} |u(x)|^p \, dx < \infty\}$. The norm on $L^p(\Omega_g)$ is $\|u\|_{L^p(\Omega_g)} = \left(\int_{\Omega_g} |u(x)|^p \, dx \right)^{\frac{1}{p}}$.
- $L^\infty(\Omega_g)$ is a Banach space. Its elements are the essentially bounded measurable functions in Ω with the norm:: $|u|_{L^\infty} := \operatorname*{ess\,sup}_{\Omega_g} |u(x)|$.
- The Sobolev space $W_p^m(\Omega_g)$; $1 \le p < \infty$; $m \ge 0$ is defined to be the set of all functions on such that for every multi-index α with $|\alpha| \le m$, the mixed partial derivative $u^{(\alpha)} = \dfrac{\partial^{|\alpha|} u}{\partial x_1^{\alpha_1} \ldots \partial x_n^{\alpha_n}}$ exists in the weak sense and is in $L^p(\Omega_g)$, i.e. $\|u\|_{W_p^m(\Omega_g)}^p = \sum_{|\alpha|=0}^{m} \int_{\Omega_g} |D^\alpha u|^p \, dx < \infty$.
- The Sobolev space $H_o^m(\Omega_g) \equiv \overset{0}{W}{}_2^m$; $m \ge 0$ defined to be the closure of the infinitely differentiable functions compactly supported in Ω_g in $H^1(\Omega_g)$. The Sobolev norm

defined above reduces here to
$\|u\|^2_{H^1(\Omega_g)} = \int\limits_{\Omega_g} |u|^2\, dx + \int\limits_{\Omega_g} |\nabla u|^2\, dx.$ When Ω_g is bounded, $\|u\|^2_{H^1_0(\Omega_g)} = \lambda \int\limits_{\Omega_g} |\nabla u|^2\, dx.$

Let $\mathbb{L}^2(\Omega, g) = (L^2(\Omega, g))^2$ and $\mathbb{H}^1_0(\Omega, g) = (H^1_0(\Omega, g))^2$ be endowed with the usual inner products and associated norms. We define

$\mathcal{V}_1 = \{u \in (C_0^\infty(\Omega, g))^2 : \nabla \cdot (gu) = 0\}$, $\mathcal{V}_2 = \{\theta \in C_0^\infty(\Omega, g)\}$,

H_g = the closure of $\mathcal{V}_1$ in $\mathbb{L}^2(\Omega, g)$, V_g = the closure of $\mathcal{V}_1$ in $\mathbb{H}^1_0(\Omega, g)$,

W_g = the closure of $\mathcal{V}_2$ in $H^1_0(\Omega, g)$, V'_g = the dual space of V_g,

W'_g = the dual space of W_g.

The inner products and norms in V_g, H_g are given by

$$(u, v)_g = \int\limits_\Omega u \cdot v g dx, \quad u, v \in H_g \text{ and } ((u, v))_g = \int\limits_\Omega \sum_{i,j=1}^{2} \nabla u_j \cdot \nabla v_i g dx, \quad u, v \in V_g$$

and norms $|u|_g^2 = (u, u)_g$, $\|u\|_g^2 = ((u, u))_g$. The norms $|\cdot|_g$ and $\|\cdot\|_g$ are equivalent to the usual ones in $\mathbb{L}^2(\Omega, g)$ and $\mathbb{H}^1_0(\Omega, g)$. We also use $\|\cdot\|_*$ for the norm in V'_g, and $\langle \cdot, \cdot \rangle$ for duality pairing between V_g and V'_g.

The inclusions $V_g \subset H_g \equiv H_{g'} \subset V_{g'}, W_g \subset L^2(\Omega, g) \subset W_{g'}$ are valid where each space is dense in the following one and the injections are continuous. By the Riesz representation theorem, it is possible to write $\langle f, u \rangle_g = (f, u)_g, \forall f \in H_g, \forall u \in V_g$.

Also, we define the orthogonal projection P_g as $P_g : H_g \to H_g$ and $\tilde{P}_g$ as $\tilde{P}_g$: $L^2(\Omega, g) \to W_g$. By taking into account the following equality

$$-\frac{1}{g}(\nabla \cdot g \nabla u) = -\Delta u - \frac{1}{g}(\nabla g \cdot \nabla) u,$$

we define the g-Laplace operator and g-Stokes operator as $-\Delta_g u = -\frac{1}{g}(\nabla \cdot g \nabla u)$ and $A_g u = P_g[-\Delta_g u]$, respectively. Since the operators A_g and P_g are self-adjoint, using integration by parts we have

$$\langle A_g u, u \rangle_g = \langle P_g[-\frac{1}{g}(\nabla \cdot g \nabla) u], u \rangle_g = \int_\Omega (\nabla u \cdot \nabla u) g dx = (\nabla u, \nabla u)_g.$$

Therefore, for $u \in V_g$, we can write $|A_g^{1/2} u|_g = |\nabla u|_g = \|u\|_g$.

Next, since the functional $\tau \in \mathrm{W}_g \mapsto (\nabla \theta, \nabla \tau)_g \in \mathbb{R}$ is a continuous linear mapping on

W_g, we can define a continuous linear mapping $\tilde{A}_g$ on $W_g^{'}$ such that

$$\forall \tau \in W_g, \langle \tilde{A}_g \theta, \tau \rangle_g = (\nabla \theta, \nabla \tau)_g, \text{ for all } \theta \in W_g.$$

We denote the bilinear operator $B_g(u,v) = P_g[(u \cdot \nabla)v]$ and the trilinear form

$$b_g(u,v,w) = \sum_{i,j=1}^{2} \int_\Omega u_i \frac{\partial v_j}{\partial x_i} w_j g dx,$$

where u, v, w lie in appropriate subspaces of V_g. Then, one obtains that $b_g(u,v,w) = -b_g(u,w,v)$, also b_g satisfies the inequality

$$b_g(u,v,v) = 0, \tag{4}$$

where $u, v, w \in V_g$.

Similarly, for $u \in V_g$ and $\theta, \tau \in W_g$ we define $\tilde{B}_g(u,\theta) = \tilde{P}_g[(u \cdot \nabla)\theta]$ and

$$\tilde{b}_g(u,\theta,\tau) = \sum_{i,j=1}^{n} \int_\Omega u_i(x) \frac{\partial \theta(x)}{\partial x_j} \tau(x) g dx.$$

Then, one obtains that $\tilde{b}_g(u,\theta,\tau) = -\tilde{b}_g(u,\tau,\theta)$ and $\tilde{b}_g$ satisfies the inequality

$$\tilde{b}_g(u,\theta,\tau) = 0, \tag{5}$$

where $u \in V_g, \theta, \tau \in W_g$.

Also b_g and $\tilde{b}_g$ satisfies the inequality (see [10],[11])

$$|b_g(u,v,w)|_g \le c_1 |u|_g^{1/2} \|u\|_g^{1/2} |v|_g |w|_g^{1/2} \|w\|_g^{1/2}, \forall u,v,w \in V_g, \tag{6}$$

$$|\tilde{b}_g(u,\theta,\tau)|_g \le c_2 |u|_g^{1/2} \|u\|_g^{1/2} |\theta|_g |\tau|_g^{1/2} \|\tau\|_g^{1/2}, \forall u \in V_g, \theta, \tau \in W_g. \tag{7}$$

We denote the operators $C_g u = P_g\Big[\frac{1}{g}(\nabla g \cdot \nabla)u\Big]$ and $\tilde{C}_g \theta = \tilde{P}_g\Big[\frac{1}{g}(\nabla g \cdot \nabla)\theta\Big]$ such that

$$\langle C_g u, v \rangle_g = b_g\left(\frac{\nabla g}{g}, u, v\right), \langle \tilde{C}_g \theta, \tau \rangle_g = \tilde{b}_g\left(\frac{\nabla g}{g}, \theta, \tau\right).$$

Finally, let $\tilde{D}_g \theta = \tilde{P}_g[\frac{\Delta g}{g}\theta]$ such that $\langle \tilde{D}_g \theta, \tau \rangle_g = -\tilde{b}_g\left(\frac{\nabla g}{g}, \theta, \tau\right) - \tilde{b}_g\left(\frac{\nabla g}{g}, \tau, \theta\right).$

Using the above notations, we can rewrite the system (1) as abstract evolutionary equations

$$\begin{cases} \dfrac{du}{dt} + B_g(u,u) + \nu A_g u + \nu C_g u = \xi\theta + f, \\ \dfrac{d\theta}{dt} + \tilde{B}_g(u,\theta) + \kappa \tilde{A}_g \theta - \kappa \tilde{C}_g \theta - \kappa \tilde{D}_g \theta = \tilde{f}, \\ u(0) = u_0, \theta(0) = \theta_0. \end{cases} \quad (8)$$

Let

$$F = \limsup_{t\to\infty} \left(\int_\Omega |f(t,x)|^2 \, dx \right)^{1/2}, \ \tilde{F} = \limsup_{t\to\infty} \left(\int_\Omega |\tilde{f}(t,x)|^2 \, dx \right)^{1/2}.$$

We define the generalized Grashof number Gr and $\tilde{G}r$ as

$$Gr = \frac{F}{\lambda_1 \nu^2}, \ \tilde{G}r = \frac{\tilde{F}}{\lambda_1 \kappa^2}.$$

The generalized Grashof number will play an analogous role as the Reynolds number and will be our bifurcation parameter. In what follows all our estimates will be in terms of the generalized Grashof number. Notice that if $f, \tilde{f}$ are time independent, then Gr and $\tilde{G}r$ are the Grashof number $G = \dfrac{|f|}{\lambda_1 \nu^2}$ and $\tilde{G} = \dfrac{|\tilde{f}|}{\lambda_1 \kappa^2}$, respectively.

In particular, we will use the following estimates (see. e.g., [1],[12]):

$$\|u\|_{L^3(\Omega)} \le c_3 |u|^{1/2} \|u\|^{1/2},$$

$$\|u\|_{L^6(\Omega)} \le c_4 \|u\|,$$

$$\|u\|_{L^\infty(\Omega)} \le c_5 \|u\|^{1/2} |Au|^{1/2}.$$

3. DETERMINING MODES

We denote by $P_m, \tilde{P}_m$ the orthogonal projection onto the linear space spanned by $\{w_1, w_2, ..., w_m\}, \{w_1', w_2', ..., w_m'\}$, the first m eigenfunctions of the Stokes operator $A, \tilde{A}$, respectively and $Q_m = I - P_m, \tilde{Q}_m = I - \tilde{P}_m$.

Let u_1, θ_1 and u_2, θ_2 solve respectively the g-Bénard problem

$$\frac{u_1}{dt} + B_g(u_1,u_1) + \nu A_g u_1 + \nu C_g u_1 = \xi\theta_1 + f_1(t), \quad (9)$$

$$\frac{\theta_1}{dt} + \tilde{B}_g(u_1,\theta_1) + \kappa \tilde{A}_g \theta_1 - \kappa \tilde{C}_g \theta_1 - \kappa \tilde{D}_g \theta_1 = \tilde{f}_1(t), \quad (10)$$

$$\frac{u_2}{dt} + B_g(u_2,u_2) + \nu A_g u_2 + \nu C_g u_2 = \xi\theta_2 + f_2(t), \quad (11)$$

$$\frac{\theta_2}{dt}+\tilde{B}_g(u_2,\theta_2)+\kappa\tilde{A}_g\theta_2-\kappa\tilde{C}_g\theta_2-\kappa\tilde{D}_g\theta_2=\tilde{f}_2(t), \tag{12}$$

where f_1, f_2 are given force in $L^\infty(0,\infty;H_g)$, and

$\tilde{f}_1, \tilde{f}_2$ are given force in $L^\infty(0,\infty;L^2(\Omega,g))$.

A set of modes $\{w_j\}_{j=1}^m$ and $\{w_j'\}_{j=1}^m$ are call determining if we have

$$\lim_{t\to\infty}|u_1(t)-u_2(t)|_g=0,$$

$$\lim_{t\to\infty}|\theta_1(t)-\theta_2(t)|_g=0,$$

whenever

$$\lim_{t\to\infty}|f_1(t)-f_2(t)|_g=0, \lim_{t\to\infty}|\tilde{f}_1(t)-\tilde{f}_2(t)|_g=0,$$

$$(13)\lim_{t\to\infty}|P_mu_1(t)-P_mu_2(t)|_g=0, \lim_{t\to\infty}|\tilde{P}_m\theta_1(t)-\tilde{P}_m\theta_2(t)|_g=0. \tag{14}$$

Lemma 3.1. [1] *Let α be a locally integrable real valued function on $(0,\infty)$, satisfying for some $0<T<\infty$ the following conditions:*

$$\liminf_{t\to\infty}\frac{1}{T}\int_t^{t+T}\alpha(\tau)d\tau=\Lambda>0, \tag{15}$$

$$\limsup_{t\to\infty}\frac{1}{T}\int_t^{t+T}\alpha^-(\tau)d\tau=\Theta<\infty, \tag{16}$$

where $\alpha^-=\max\{-\alpha,0\}$. Further, let β be a real valued locally integrable function defined on $(0,\infty)$ such that

$$\lim_{t\to\infty}\frac{1}{T}\int_t^{t+T}\beta^+(\tau)d\tau=0, \textit{where } \beta^+=\max\{\beta,0\}.$$

Suppose that ζ is an absolutely continuous non-negative function on $(0,\infty)$ such that

$$\frac{d}{dt}\zeta+\alpha\zeta\le\beta, \textit{a.e. on } (0,\infty).$$

Then $\zeta(t)\to 0$ as $t\to\infty$.

Lemma 3.2. *Let $(u(t),\theta(t))$ are solve of the g-Bénard problem (1). Then the estimates*

$$\limsup_{t\to\infty}\frac{1}{T}\int_t^{t+T}\|u\|_g^2d\tau\le c_7Gr^2+c_8\tilde{G}r^2, \tag{17}$$

$$\limsup_{t\to\infty}\frac{1}{T}\int_t^{t+T}\|\theta\|_g^2d\tau\le c_6\tilde{G}r^2. \tag{18}$$

for every $t,T>0$.

Proof. Multiplying the first equation of (1) by u and the second by θ, we get

$$\frac{1}{2}\frac{d}{dt}|u|_g^2+\nu\|u_g^2\|+\nu b_g(\frac{\nabla g}{g},u,u)=(\xi\theta,u)_g+(f,u)_g,$$
$$\frac{1}{2}\frac{d}{dt}|\theta|_g^2+\kappa\|\theta_g^2\|+\kappa\tilde{b}_g(\frac{\nabla g}{g},\theta,\theta)=(\tilde{f},\theta)_g.$$

By using the properties of the trilinear from b_g and $\tilde{b}_g$, the Cauchy-Schwarz and the Young inequalities, we see that

$$\frac{d}{dt}|u|_g^2+\nu\gamma_0\|u_g^2\|\le\frac{2|\xi|_\infty^2}{\gamma_0\nu}|\theta|_g^2+\frac{2}{\gamma_0\nu\lambda_1}|f|_g^2, \tag{19}$$

$$\frac{d}{dt}|\theta|_g^2+\kappa\gamma_0\|\theta\|_g^2\le\frac{1}{\gamma_0\kappa\lambda_1}|\tilde{f}|_g^2, \tag{20}$$

where $\gamma_0=\left(1-\frac{|\nabla g|_\infty}{m_0\lambda_1^{1/2}}\right)>0$.

Using $\|\theta\|_g^2\ge\lambda_1|\theta|_g^2$ for (20), we have

$$\frac{d}{dt}|\theta|_g^2+\gamma_0\kappa\lambda_1|\theta|_g^2\le\frac{1}{\gamma_0\kappa\lambda_1}|\tilde{f}|_g^2 .$$

Now from Gronwall's inequality we obtain

$$|\theta(t)|_g^2\le e^{-\gamma_0\kappa\lambda_1 t}|\theta(0)|_g+\frac{1}{\gamma_0\kappa\lambda_1}\int_0^t e^{-\gamma_0\kappa\lambda_1(t-s)}|\tilde{f}(s)|_g^2\,ds.$$

In particular, we have

$$\limsup_{t\to\infty}|\theta(t)|_g^2\le\frac{\tilde{F}^2}{\gamma_0^2\kappa^2\lambda_1^2}. \tag{21}$$

We integrate (20) over the interval $(t,t+T)$, we obtain

$$\gamma_0\kappa\int_t^{t+T}\|\theta\|_g^2d\tau\le|\theta(t)|_g^2+\frac{1}{\gamma_0\kappa\lambda_1}\int_t^{t+T}|\tilde{f}|_g^2\,d\tau.$$

That is

$$\limsup_{t\to\infty}\frac{1}{T}\int_t^{t+T}\|\theta\|_g^2d\tau\le\frac{\tilde{F}^2}{T\gamma_0^3\kappa^3\lambda_1^2}+\frac{\tilde{F}^2}{\gamma_0^2\kappa^2\lambda_1}.$$

From $\tilde{G}r=\frac{\tilde{F}}{\lambda_1\kappa^2}$ and take $T=(\gamma_0\gamma\lambda_1)^{-1}$, we obtain

$$\limsup_{t\to\infty}\frac{1}{T}\int_t^{t+T}\|\theta\|_g^2d\tau\le\frac{2\gamma^2\lambda_1}{\gamma_0^2}\tilde{G}r^2,$$

where $\gamma = \max\{\nu, \kappa\}$ and set $c_6 = \dfrac{2\gamma^2\lambda_1}{\gamma_0^2}$, we have estimate (18).

Next, using $\| u \|_g^2 \geq \lambda_1 | u |_g^2$ and Gronwall's inequality for (19) we obtain

$$| u(t) |_g^2 \leq e^{-\gamma_0 \nu \lambda_1 t} | u(0) |_g^2 + \frac{2}{\gamma_0 \nu \lambda_1} \int_0^t e^{-\gamma_0 \nu \lambda_1 (t-s)} (| \xi |_\infty^2 | \theta |_g^2 + | f |_g^2) ds.$$

In particular, we have

$$\limsup_{t\to\infty} | u(t) |_g^2 \leq \frac{2 | \xi |_\infty^2 \tilde{F}^2}{\gamma_0^4 \nu^2 \kappa^2 \lambda_1^4} + \frac{2F^2}{\gamma_0^2 \nu^2 \lambda_1^2}.$$

Using $| \theta |_g^2 \leq \dfrac{1}{\lambda_1} \| \theta \|_g^2$ for (19), then we integrate it over the interval $(t, t+T)$, we obtain

$$\int_t^{t+T} \| u \|_g^2 d\tau \leq \frac{1}{\gamma_0 \nu} | u(t) |_g^2 + \frac{2 | \xi |_\infty^2}{\gamma_0^2 \nu^2 \lambda_1} \int_t^{t+T} \| \theta \|_g^2 d\tau + \frac{2}{\gamma_0^2 \nu^2 \lambda_1} \int_t^{t+T} | f |_g^2 \, d\tau.$$

That is

$$\begin{aligned} &\limsup_{t\to\infty} \frac{1}{T} \int_t^{t+T} \| u \|_g^2 d\tau \\ &\leq \frac{2 | \xi |_\infty^2 \tilde{F}^2}{T\gamma_0^5 \nu^3 \kappa^2 \lambda_1^4} + \frac{2F^2}{T\gamma_0^3 \nu^3 \lambda_1^2} + \frac{2 | \xi |_\infty^2 \tilde{F}^2}{T\gamma_0^5 \nu^2 \kappa^3 \lambda_1^3} + \frac{2 | \xi |_\infty^2 \tilde{F}^2}{\gamma_0^4 \nu^2 \kappa^2 \lambda_1^2} + \frac{2\nu^2 \lambda_1 F^2}{\gamma_0^2}. \end{aligned}$$

From $Gr = \dfrac{F}{\lambda_1 \nu^2}$ and $\tilde{G}r = \dfrac{\tilde{F}}{\lambda_1 \kappa^2}$, we obtain

$$\begin{aligned} &\limsup_{t\to\infty} \frac{1}{T} \int_t^{t+T} \| u \|_g^2 d\tau \\ &\leq \frac{2\kappa^2 | \xi |_\infty^2 \tilde{G}r^2}{T\gamma_0^5 \nu^3 \lambda_1^2} + \frac{2\nu Gr^2}{T\gamma_0^3 \nu^3} + \frac{2\kappa | \xi |_\infty^2 \tilde{G}r^2}{T\gamma_0^5 \nu^2 \lambda_1} + \frac{2\kappa^2 | \xi |_\infty^2 \tilde{G}r^2}{\gamma_0^4 \nu^2} + \frac{2\nu^2 \lambda_1 Gr^2}{\gamma_0^2}. \end{aligned}$$

Setting $c_7 = \dfrac{4\gamma^2 \lambda_1}{\gamma_0^2}, c_8 = \left(\dfrac{2\gamma^3 | \xi |_\infty^2}{\gamma_0^4 \nu^3 \lambda_1} + \dfrac{4\gamma^2 | \xi |_\infty^2}{\gamma_0^4 \nu^2} \right)$ we obtain estimate (17).

Theorem 3.1. *Suppose that* m *satisfies*

$$\lambda_{m+1} \geq c_9 Gr^2 + c_{10} \tilde{G}r^2 + c_{11},$$

$$\lambda_{m+1} \geq c_{12} \tilde{G}r^4 + c_{13}.$$

Then the number of determining modes is not larger than m. That is, if

$$\lim_{t\to\infty} | P_m u_1(t) - P_m u_2(t) |_g = 0,$$

$$\lim_{t\to\infty} | \tilde{P}_m \theta_1(t) - \tilde{P}_m \theta_2(t) |_g = 0,$$

then

$$\lim_{t\to\infty} |u_1(t)-u_2(t)|_g = 0, \lim_{t\to\infty} |\theta_1(t)-\theta_2(t)|_g = 0.$$

Proof. Let

$$w(t) = u_1(t) - u_2(t),\ p(t) = P_m w(t) \text{ and } q(t) = Q_m w(t),$$
$$\tilde{w}(t) = \theta_1(t) - \theta_2(t),\ \tilde{p}(t) = \tilde{P}_m \tilde{w}(t) \text{ and } \tilde{q}(t) = \tilde{Q}_m \tilde{w}(t).$$

Then by assumption $|p| \to 0$ and $|\tilde{p}| \to 0$ as $t \to \infty$.

Subtracting Equation (11) from (9) and (12) from (10), we obtain

$$\frac{dw}{dt} + \nu A_g w + \nu C_g w + B_g(u_1, w) + B_g(w, u_1) - B_g(w, w) = \xi\, \tilde{w} + f_1(t) - f_2(t), \tag{22}$$

$$\frac{d\tilde{w}}{dt} + \kappa \tilde{A}_g \tilde{w}\theta - \kappa \tilde{C}_g \tilde{w} - \kappa \tilde{D}_g \tilde{w} + \tilde{B}_g(u_1, \tilde{w}) + \tilde{B}_g(w, \theta_1) - \tilde{B}_g(w, \tilde{w}) = \tilde{f}_1(t) - \tilde{f}_2(t). \tag{23}$$

Taking the inner product (22) with q and (23) with $\tilde{q}$. Then, using (4) and (5), we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}|q|_g^2 + \nu \|q\|_g^2 \le& |(B_g(w,u_1), w)| + |(B_g(u_1, w) + B_g(w, u_1) - B_g(w, w), p)| \\ &+ \nu |b_g(\frac{\nabla g}{g}, w, q)| + |(\xi \tilde{w}, q)| + |f_1(t) - f_2(t)|_g |q|_g,\end{aligned} \tag{24}$$

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}|\tilde{q}|_g^2 + \kappa \|\tilde{q}\|_g^2 \le& |(\tilde{B}_g(w,\theta_1), \tilde{w})| + |(B_g(u_1, \tilde{w}) + \tilde{B}_g(w, \theta_1) - \tilde{B}_g(w, \tilde{w}), \tilde{p})| \\ &+ \kappa |\tilde{b}_g(\frac{\nabla g}{g}, \tilde{q}, \tilde{w})| + |\tilde{f}_1(t) - \tilde{f}_2(t)|_g |\tilde{q}|_g .\end{aligned} \tag{25}$$

By using the (6), Cauchy-Schwarz inequality and Young's inequality we give some bounds on the terms which occur in the (24)

$$\begin{aligned}&|(B_g(w,u_1), w)| \\ &= |(B_g(q,u_1), q)| + |(B_g(q,u_1), p)| + |(B_g(p,u_1), w)| \\ &\le c_1 |q|_g \|q\|_g \|u_1\|_g + |p|_g^{1/2} (c_1 |q|_g^{1/2} \|q\|_g^{1/2} \|u_1\|_g \|p\|_g^{1/2} + c_1 \|p\|_g^{1/2} \|u_1\|_g |w|_g^{1/2} \|w\|_g^{1/2}) \\ &\le \frac{3c_1^2}{2\nu} |q|_g^2 \|u_1\|_g^2 + \frac{\nu}{6} \|q\|_g^2 \\ &\quad + |p|_g^{1/2} (c_1 |q|_g^{1/2} \|q\|_g^{1/2} \|u_1\|_g \|p\|_g^{1/2} + c_1 \|p\|_g^{1/2} \|u_1\|_g |w|_g^{1/2} \|w\|_g^{1/2}) \\ &\qquad := \frac{3c_1^2}{2\nu} |q|_g^2 \|u_1\|_g^2 + \frac{\nu}{6} \|q\|_g^2 + M_1(t) |p|_g^{1/2},\end{aligned} \tag{26}$$

$$\begin{aligned}&|(B_g(u_1, w) + B_g(w, u_1) - B_g(w, w), p)| \\ &\le |(B_g(u_1, w), p)| + |(B_g(w, u_1), p)| + |(B_g(w, w), p)|\end{aligned}$$

$$\leq |p|_g^{1/2}\,(c_1|u_1|_g^{1/2}\|u_1\|_g^{1/2}\|w\|_g\|p\|_g^{1/2}$$

$$+c_1|w|_g^{1/2}\|w\|_g^{1/2}\|u_1\|_g\|p\|_g^{1/2}+c_1|w|_g^{1/2}\|w\|_g^{3/2}\|p\|_g^{1/2}) := M_2(t)|p|_g^{1/2}, \tag{27}$$

$$|b_g(\frac{\nabla g}{g},w,q)| \leq |b_g(\frac{\nabla g}{g},q,q)|+|b_g(\frac{\nabla g}{g},p,q)| \leq \frac{|\nabla g|_\infty}{m_0}\|q\|_g|q|_g+\frac{|\nabla g|_\infty}{m_0}\|p\|_g|q|_g$$

$$\leq \frac{3|\nabla g|_\infty^2}{2m_0^2}|q|_g^2+\frac{1}{6}\|q\|_g{}^2+\lambda_m^{1/2}\frac{|\nabla g|_\infty}{m_0}|p|_g|q|_g$$

$$:= \frac{3|\nabla g|_\infty^2}{2m_0^2}|q|_g^2+\frac{1}{6}\|q\|_g{}^2+M_3(t)|p|_g, \tag{28}$$

$$|(\xi\tilde{w},q)| \leq |(\xi\,\tilde{q},q)|+|(\xi\tilde{p},q)| \leq |\xi|_\infty|\tilde{q}|_g|q|_g+|\xi|_\infty|\tilde{p}|_g|q|_g$$

$$\leq \frac{1}{2}|\xi|_\infty^2|\tilde{q}|_g^2+\frac{1}{2}|q|_g^2+|\xi|_\infty|\tilde{p}|_g|q|_g\,. \tag{29}$$

Using estimations (26)-(28) and $\|q^2\| \geq \lambda_{m+1}|q|^2$ into (24), we infer that

$$\frac{d}{dt}|q|_g^2+|q|_g^2\left[\nu\lambda_{m+1}-\left(\frac{3c_1^2}{\nu}\|u_1\|_g^2+\frac{3\nu|\nabla g|_\infty^2}{m_0^2}+1\right)\right]+\frac{\nu}{3}\|q\|_g^2$$

$$\leq 2M_1(t)|p|_g^{1/2}+2M_2(t)|p|_g^{1/2}+2\nu M_3(t)|p|+|\xi|_\infty^2|\tilde{q}|_g^2$$

$$+2|\xi|_\infty|\tilde{p}|_g|q|_g+2|f_1(t)-f_2(t)|_g|q|_g\,. \tag{30}$$

Next, using the (7), Cauchy-Schwarz inequality and Young's inequality we give some bounds on the terms which occur in the (25)

$$|(\tilde{B}_g(w,\theta_1),\tilde{w})|$$

$$=|(\tilde{B}_g(q,\theta_1),\tilde{q})|+|(\tilde{B}_g(q,\theta_1),\tilde{p})|+|(\tilde{B}_g(p,\theta_1),\tilde{w})|$$

$$\leq c_2|q|_g^{1/2}\|q_g^{1/2}\|_g\|\theta_1\|_g|\tilde{q}|_g^{1/2}\|\tilde{q}\|_g^{1/2}+c_2|q|_g^{1/2}\|q\|_g^{1/2}\|\theta_1\|_g|\tilde{p}|_g^{1/2}\|\tilde{p}\|_g^{1/2}$$

$$+c_2|p|_g^{1/2}\|p\|_g^{1/2}\|\theta_1\|_g|\tilde{w}|_g^{1/2}\|\tilde{w}\|_g^{1/2}$$

$$:= c_2|q|_g^{1/2}\|q\|_g^{1/2}\|\theta_1\|_g|\tilde{q}|_g^{1/2}\|\tilde{q}\|_g^{1/2}+N_1(t)|\tilde{p}|_g^{1/2}+N_2(t)|p|_g^{1/2},$$

$$\leq |q|_g\|q\|_g+\frac{c_2^2}{4}\|\theta_1\|_g^2|\tilde{q}|_g\|\tilde{q}\|_g+N_1(t)|\tilde{p}|_g^{1/2}+N_2(t)|p|_g^{1/2}$$

$$\leq \frac{3}{2\nu}|q|_g^2+\frac{\nu}{6}\|q\|_g^2+\frac{c_2^4}{16\kappa}\|\theta_1\|_g^4|\tilde{q}|_g^2+\frac{\kappa}{4}\|\tilde{q}\|_g^2+N_1(t)|\tilde{p}|_g^{1/2}+N_2(t)|p|_g^{1/2}, \tag{31}$$

$$|(\tilde{B}_g(u_1,\tilde{w})+\tilde{B}_g(w,\theta_1)-\tilde{B}_g(w,\tilde{w}),\tilde{p})|$$

$$\leq |(\tilde{B}_g(u_1,\tilde{w}),\tilde{p})|+|(\tilde{B}_g(w,\theta_1),\tilde{p})|+|(\tilde{B}_g(w,\tilde{w}),\tilde{p})|$$

$$\leq |\tilde{p}|_g^{1/2} (c_2 |u_1|_g^{1/2} \|u_1\|_g^{1/2} \|\tilde{w}\|_g \|\tilde{p}\|_g^{1/2}$$

$$+c_2 |w|_g^{1/2} \|w\|_g^{1/2} \|\theta_1\|_g \|\tilde{p}\|_g^{1/2} + c_2 |w|_g^{1/2} \|w\|_g^{1/2} \|\tilde{w}\|_g \|\tilde{p}\|_g^{1/2})$$

$$:= N_3(t) |\tilde{p}|_g^{1/2}, \tag{32}$$

$$|\tilde{b}_g(\frac{\nabla g}{g}, \tilde{q}, \tilde{w})| \leq |\tilde{b}_g(\frac{\nabla g}{g}, \tilde{q}, \tilde{q})| + |\tilde{b}_g(\frac{\nabla g}{g}, \tilde{q}, \tilde{p})| \leq \frac{|\nabla g|_\infty}{m_0} \|\tilde{q}\|_g |\tilde{q}|_g + \frac{|\nabla g|_\infty}{m_0} \|\tilde{q}\|_g |\tilde{p}|_g$$

$$\leq \frac{|\nabla g|_\infty^2}{m_0^2} |\tilde{q}|_g^2 + \frac{1}{4} \|\tilde{q}\|_g^2 + \frac{|\nabla g|_\infty}{m_0} \|\tilde{q}\|_g |\tilde{p}|_g$$

$$:= \frac{|\nabla g|_\infty^2}{m_0^2} |\tilde{q}|_g^2 + \frac{1}{4} \|\tilde{q}\|_g^2 + N_4(t) |\tilde{p}|_g \, . \tag{33}$$

Using estimations (31)-(33) and $\|\tilde{q}\|_g^2 \geq \lambda_{m+1} |\tilde{q}|_g^2$ into (24), we infer that

$$\frac{d}{dt} |\tilde{q}|_g^2 + |\tilde{q}|_g^2 \left[\kappa\lambda_{m+1} - \left(\frac{c_2^4}{8\kappa} \|\theta_1\|_g^4 + \frac{2\kappa |\nabla g|_\infty^2}{m_0^2} \right) \right]$$

$$\leq \frac{3}{\nu} |q|_g^2 + \frac{\nu}{3} \|q\|_g^2 + 2N_1(t) |\tilde{p}|_g^{1/2} + 2N_2(t) |p|_g^{1/2} + 2N_3(t) |\tilde{p}|_g^{1/2}$$

$$+2\kappa N_4(t) |\tilde{p}|_g + 2 |\tilde{f}_1(t) - \tilde{f}_2(t) \| \tilde{q}|_g \, . \tag{34}$$

We sum equations (30) and (34) to obtain

$$\frac{d}{dt}\zeta + \alpha(t)\zeta \leq \beta(t),$$

where

$$\zeta = |q|_g^2 + |\tilde{q}|_g^2, \; \alpha(t) = \min\{\alpha_1(t), \alpha_2(t)\},$$

$$\alpha_1(t) = \nu\lambda_{m+1} - \left(\frac{3c_1^2}{\nu} \|u_1\|_g^2 + \frac{3\nu |\nabla g|_\infty^2}{m_0^2} + \frac{3}{\nu} + 1 \right),$$

$$\alpha_2(t) = \kappa\lambda_{m+1} - \left(\frac{c_2^4}{8\kappa} \|\theta_1\|_g^4 + \frac{2\kappa |\nabla g|_\infty^2}{m_0^2} + |\xi|_\infty^2 \right),$$

$$\beta(t) = 2M_1(t) |p|_g^{1/2} + 2M_2(t) |p|_g^{1/2} + 2\nu M_3(t) |p| + 2 |\xi|_\infty |\tilde{p}|_g |q|_g$$

$$+2 |f_1(t) - f_2(t)|_g |q|_g + 2N_1(t) |\tilde{p}|_g^{1/2} + 2N_2(t) |p|_g^{1/2}$$

$$+2N_3(t) |\tilde{p}|_g^{1/2} + 2\kappa N_4(t) |\tilde{p}|_g + 2 |\tilde{f}_1(t) - \tilde{f}_2(t) \| \tilde{q}|_g \, .$$

Since the solutions u_1, u_2, θ_1 and θ_2 are bounded uniformly for t bounded away from zero in H_g , V_g and W_g respectively and by assumptions (13) and (14) it follows that $\beta(0) \to 0$ as $t \to \infty$

From (17) and (18), we see that

$$\limsup_{t\to\infty} \frac{1}{T}\int_t^{t+T} \alpha_1^-(\tau)d\tau \le \frac{3c_1^2}{\nu}(c_7 Gr^2 + c_8\tilde{G}r^2) + \frac{3\nu|\nabla g|_\infty^2}{m_0^2} + \frac{3}{\nu} + 1 - \nu\lambda_{m+1} < \infty,$$

$$\limsup_{t\to\infty} \frac{1}{T}\int_t^{t+T} \alpha_2^-(\tau)d\tau \le \frac{c_2^4}{8\kappa}c_6^2\tilde{G}r^4 + \frac{2\kappa|\nabla g|_\infty^2}{m_0^2} + |\xi|_\infty^2 - \kappa\lambda_{m+1} < \infty,$$

and the condition (16) of Lemma 3.1 is satisfied.

Finally, we see that

$$\liminf_{t\to\infty} \frac{1}{T}\int_t^{t+T} \alpha_1(\tau)d\tau \ge \nu\lambda_{m+1} - \left(\frac{3c_1^2}{\nu}(c_7 Gr^2 + c_8\tilde{G}r^2) + \frac{3\nu|\nabla g|_\infty^2}{m_0^2} + \frac{3}{\nu} + 1\right),$$

$$\liminf_{t\to\infty} \int_t^{t+T} \alpha_2(\tau)d\tau \ge \kappa\lambda_{m+1} - \left(\frac{c_2^4}{8\kappa}c_6^2\tilde{G}r^4 + \frac{2\kappa|\nabla g|_\infty^2}{m_0^2} + |\xi|_\infty^2\right),$$

and if m is sufficiently large that the inequalities

$$\lambda_{m+1} \ge \frac{3c_1^2}{\nu^2}(c_7 Gr^2 + c_8\tilde{G}r^2) + \frac{3|\nabla g|_\infty^2}{m_0^2} + \frac{3}{\nu^2} + \frac{1}{\nu},$$

$$\lambda_{m+1} \ge \frac{c_2^4}{8\kappa^2}c_6^2\tilde{G}r^4 + \frac{2|\nabla g|_\infty^2}{m_0^2} + \frac{|\xi|_\infty^2}{\kappa}.$$

That is

$$\lambda_{m+1} \ge c_9 Gr^2 + c_{10}\tilde{G}r^2 + c_{11},\ \lambda_{m+1} \ge c_{12}\tilde{G}r^4 + c_{13},$$

where

$$c_9 := \frac{3c_1^2c_7}{\nu^2}, c_{10} := \frac{3c_1^2c_8}{\nu^2}, c_{11} := \frac{3|\nabla g|_\infty^2}{m_0^2} + \frac{3}{\nu^2} + \frac{1}{\nu}, c_{12} := \frac{c_2^4c_6^2}{8\kappa^2}, c_{13} := \frac{2|\nabla g|_\infty^2}{m_0^2} + \frac{|\xi|_\infty^2}{\kappa}.$$

Hence, from the above and Lemma 3.1 we conclude that:

$$\lim_{t\to\infty}|u_1(t) - u_2(t)|_g = 0 \text{ and } \lim_{t\to\infty}|\theta_1(t) - \theta_2(t)|_g = 0.$$

4. CONCLUSION

In conclusion, we have presented an improved upper bound on the number of modes defined for the 2D g-Bénard problem. Moreover, this is an important result in the study on the long-time behavior of the solution when the time to infinity. The calculation techniques showed here are able to be applied to other classes of equation systems such as: Boussinesq and MHD.

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