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ALGEBRA OF POLYNOMIALS BOUNDED ON SOME STRIPS

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Abstract. In the present paper, for a finite sequence of single variable polynomials $g(x)=(g_1(x), g_2(x),..., g_n(x))$, we study the ring of geometrically bounded elements on a generalized strip $M_c(g)$ in $\mathbb{R}^{n+1}$ which is the solution of the system of polynomial inequalities $g_1(x)\le y_1 \le g_1(x)+c_1$, $g_2(x)\le y_2 \le g_2(x)+c_2$ …, $g_n(x)\le y_n \le g_n(x)+c_n$. This ring is shown to be the finitely generated $\mathbb{R}-$algebra $\mathbb{R}[y_1\text{-}g_1(x), y_2\text{-}g_2(x),..., y_n\text{-}g_n(x)]$ provided that $c=(c_1,c_2,...,c_n)$ is a positive vector. However, if $c=0$ then this algebra is not finitely generated in general. In particular, we point out that the ring of geometrically bounded elements on 'a generalized strip' of the form $M(g_1,g_2)$ in $\mathbb{R}^2$ which is the solution of the polynomial inequality $g_1(x)\le y\le g_2(x)$ is trivial (i.e., is equal to $\mathbb{R}$) provided that $g_1(x)$ is less than $g_2(x)$ at infinity. As a consequence, we can describe the ring of geometrically bounded elements on a finite union of disjoint strips.

Keywords: Positivstellensatze, bounded polynomials, semi-algebraic.

1. INTRODUCTION

Starting with the 17^{th} Hilbert's Problem, many problems have arisen in Real Algebraic Geometry, and many interesting results have been obtained. Given a basic closed semi-algebraic set $K=\{y\in\mathbb{R}^n : f_1(y)\ge 0, f_2(y)\ge 0,..., f_m(y)\ge 0\}$, where $f_1, f_2,..., f_m$ are real polynomials. Denominator-free Positivstellensatze are results characterizing all polynomials, which are positive on K in terms of sums of squares and the polynomials f_i used to describe K. Theorems about the existence of such representations have various applications, notably in problems of optimizing polynomial functions on semi-algebraic sets. For a survey and details, we refer the reader to [1-5] and the references therein.

In the case K is compact, Schmudgen [6] has proved that any polynomial, which is positive on K, is in the pre-ordering $T = T(f_1, f_2, ..., f_m)$ (T is the set of finite sums of elements in the form $s_e f_1^{e_1} f_2^{e_2} ... f_m^{e_m}$, where $e_i \in \{0,1\}$ and each s_e is a sum of squares of polynomials). For non-compact case, there are some results such as [7, 8], etc.

Note that Schweighofer [9] gave an algebraic proof of Schmudgen Positivstellensatze [6, Corollary 3]. In this proof, the bounded algebra is an important key. (From that proof, many works studying the algebra of polynomials bounded on a semi-algebraic set have been published such as [10-12], etc).

The strip $\mathbb{R}\times[0,1]$ was studied in [13] (or also in [14]) which stated that a real polynomial which is nonnegative on a strip $\mathbb{R}\times[0,1]$ belongs to the pre-ordering $T\left(y(1-y)\right)$. Replacing the horizontal lines $y=0, y=1$ by $y_i = g_i(x), y_i = g_i(x)+c_i$, we define a generalized strip as follows:

$$M_c(g) = M_c\left(g_1,...,g_n\right) := \left\{(x,y)\in\mathbb{R}^{1+n} : g_i(x) \le y_i \le g_i(x)+c_i; \forall i = 1,2...,n\right\},$$

where $g_i(x)$ are real single variable polynomials and $c=(c_1,...,c_n)$ is a vector of positive coordinates. When $n=1, g\equiv 0$ and $c=1$ the strip $M_1(0)=\mathbb{R}\times[0,1]$ is defined as in [13]. In this paper, we calculate the algebra of polynomials which are bounded on the generalized strips $M_c\left(g_1,...,g_n\right)$. It turns out that the algebra mentioned here is finitely generated (Theorem 2.1). We recall that $\mathbb{R}$ - algebra A is said to be *finitely generated* if there exists a finite set $U=\{u_1,u_2,...,u_k\}$ such that $A=\mathbb{R}[U]=\mathbb{R}[u_1,u_2,...,u_k]$.

However, when $c=0$ and $n=1$ the generalized strip $M_0(g)$ is the curve $y=g(x)$, and the algebra of polynomials bounded on $y=g(x)$ is not finitely generated in general (Corollary 2.2).

2. MAIN RESULTS

Throughout this paper, $\mathbb{Z}$ denotes the set of integer numbers, $\mathbb{N}$ the set of positive integer numbers, and $\mathbb{R}^n$ the Euclidean space of dimension n. The ring of real polynomials in n variables $y_1,...,y_n$ is denoted by $\mathbb{R}[y]=\mathbb{R}[y_1,...,y_n]$ and the ring of real polynomials in single variable x is denoted by $\mathbb{R}[x]$. For a subset K of $\mathbb{R}^n$, by $B(K)$ we denote the set of all polynomials bounded on K. Then $B(K)$ is called the bounded algebra and is a subalgebra of $\mathbb{R}[y]$ over $\mathbb{R}$.

We start with the bounded algebra on a compact cross-section set.

Lemma 2.1. *Let $n\in\mathbb{N}$ and K be a compact subset of $\mathbb{R}^n$. Suppose that*

$$K \setminus p^{-1}(0) \neq \varnothing;\ \forall p \in \mathbb{R}[y]\setminus\{0\}. \tag{1}$$

Then the algebra of polynomials in $\mathbb{R}[x, y_1, y_2, ..., y_n]$ *which are bounded on* $\mathbb{R}\times K$ *is* $\mathbb{R}[y] = \mathbb{R}[y_1, y_2, ..., y_n]$.

Proof:

Since K is compact, we immediately get $\mathbb{R}[y] \subseteq B(\mathbb{R}\times K)$. We prove the converse direction. For $f(x, y) \in B(\mathbb{R}\times K)$, we can write

$$f(x, y) = \sum_{i=0}^{m} f_i(y)x^i. \tag{2}$$

Assume that there is an index $i > 0$ such that $f_i \neq 0$. Since K has the property (1), there exists $y_0 \in K$ such that $f_i(y_0) \neq 0$. Hence, the single variable polynomial $f(x, y_0)$ is of degree at least $i > 0$ and so it cannot be bounded on $\mathbb{R}$. This is a contradiction. Therefore, $f_i = 0, \forall i > 0$ and $f(x, y) = f_0(y)$, i.e., $f(x, y) \in \mathbb{R}[y]$.

Hence, we have $B(\mathbb{R}\times K) = \mathbb{R}[y]$.

Corollary 2.1. *Let* $n \in \mathbb{N}$ *and* K *be a compact subset of* $\mathbb{R}^n$. *If the interior of* K *is non-empty, then the algebra of polynomials in* $\mathbb{R}[x, y_1, y_2, ..., y_n]$ *which are bounded on* $\mathbb{R}\times K$ *is* $\mathbb{R}[y_1, y_2, ..., y_n]$.

Proof:

Since the interior of K is non-empty, K contains an open ball and so the dimension of K is *n.* On the other hand, for all non-zero polynomial $p \in \mathbb{R}[y_1, y_2, ..., y_n]$, the hypersurface $p^{-1}(0)$ is of dimension $n-1$. Hence, $K \setminus p^{-1}(0) \neq \varnothing$ and thus the property (1) holds. Therefore $B(\mathbb{R}\times K) = \mathbb{R}[y]$.

Let $g(x) = (g_1(x), g_2(x), ..., g_n(x))$ be a vector of n polynomials in the single variable ring $\mathbb{R}[x]$ and $c = (c_1, c_2, ..., c_n)$ a vector of positive numbers. A generalized strip $M_c(g)$ is a closed basic semi-algebraic set defined by

$$M_c(g) := \left\{(x, y) \in \mathbb{R}^{1+n} : g_i(x) \leq y_i \leq g_i(x) + c_i; \forall i = 1, 2..., n\right\}. \tag{3}$$

Theorem 2.1. *Let* g, c, $M_c(g)$ *be defined as above. Then the algebra of polynomials bounded on* $M_c(g)$ *is generated by* $y_1 - g_1(x),\ y_2 - g_2(x), ...,\ y_n - g_n(x)$, *i.e.,*

$$B(M_c(g)) = \mathbb{R}[y_1 - g_1(x), y_2 - g_2(x), ..., y_n - g_n(x)]. \tag{4}$$

Proof:

Performing change of variables $z_i = y_i - g_i(x)$ for all $i = \overline{1, n}$, we have $(x, y) \in M_c(g)$ if and only if $(x, z) \in \mathbb{R}\times[0, c]$, where $[0, c] = [0, c_1]\times...\times[0, c_n]$ and $z = (z_1, ..., z_n)$. Applying Corollary 2.1 with $K = [0, c]$, we get that a polynomial $p(x, z) \in \mathbb{R}[x, z]$ is bounded on $\mathbb{R}\times[0, c]$ if and only if $p(x, z) \in \mathbb{R}[z]$. Hence, a polynomial $f(x, y)$ is bounded on $M_c(g)$ if and only if $f(x, z+g(x))$ is bounded on $\mathbb{R}\times[0,c]$, and so if and only if there exists $p(z)\in\mathbb{R}[z]$ such that $f(x,z+g(x))=p(z)$. That means

$$f(x,y)=p(y-g(x))=p\left(y_1-g_1(x),...,y_n-g_n(x)\right).$$

Remark 2.1: If we replace the strip $M_c(g)$ in Theorem 2.1 by the corresponding half strip:

$$M_c(g)\cap\left\{(x,y)\in\mathbb{R}^{1+n}: x\geq x_0\right\}, \tag{5}$$

then the algebra of polynomials bounded on the half strip is the same as the one on the strip $M_c(g)$. The proof follows similarly from the proof of Theorem 2.1, since the algebra of polynomials bounded on $[x_0,+\infty]\times[0,c]$ is the same as that on $\mathbb{R}\times[0,c]$. Indeed, applying Corollary 2.1 for $K=[0,c]$ we have that the algebra of polynomials in $\mathbb{R}[x,y_1,y_2,...,y_n]$ bounded on $\mathbb{R}\times[0,c]$ is $\mathbb{R}[y_1,y_2,...,y_n]$. By a similar argument as in the proof of Lemma 2.1 we also get that the algebra of polynomials bounded on $[x_0,+\infty]\times K$ is $\mathbb{R}[y_1,y_2,...,y_n]$, where K has Property (1). Applying this result for $K=[0,c]$ we find that $B\left([x_0,+\infty]\times[0,c]\right)$ is also $\mathbb{R}[y_1,y_2,...,y_n]$.

For any single variable polynomials $g_1(x), g_2(x)$, 'a generalized strip' is defined in the following form:

$$M(g_1,g_2):=\left\{(x,y)\in\mathbb{R}^2: g_1(x)\leq y\leq g_2(x)\right\}. \tag{6}$$

We say $g_1<g_2$ *at infinity* if $\lim\limits_{x\to+\infty}\left(g_2(x)-g_1(x)\right)=+\infty$ or $\lim\limits_{x\to-\infty}\left(g_2(x)-g_1(x)\right)=+\infty$. In the following proposition, we show that the algebra of bounded polynomials becomes trivial on $M(g_1,g_2)$ if $g_1<g_2$ at infinity.

Proposition 2.1. *Let $M(g_1,g_2)$ be defined as above. Suppose that $g_1<g_2$ at infinity. Then the algebra of polynomials bounded on $M(g_1,g_2)$ is $\mathbb{R}$.*

In order to prove Proposition 2.1, we need the following lemma.

We denote by $cone\left(v_1,v_2,...,v_m\right)$ the convex cone finitely generated by $v_1,v_2,...,v_m$ in $\mathbb{R}^n$, i.e.,

$$cone\left(v_1,v_2,...,v_m\right):=\left\{\sum_{i=1}^{m}\lambda_i v_i:\lambda_i\geq 0;\forall i=1,2,...,m\right\}. \tag{7}$$

In this paper, we only consider finitely generated convex cones. The dimension of the cone $cone\left(v_1,v_2,...,v_m\right)$ is defined to be the dimension of the $span\left(v_1,v_2,...,v_m\right)$. Suppose that C is a cone. We denote by $C_v:=v+C$ the translation of C by $v\in\mathbb{R}^n$. The dimension of C_v is defined to be the dimension of C.

Lemma 2.2. *Let* $n \in \mathbb{N}$ *and* K *be a subset of* $\mathbb{R}^n$. *If* K *contains an* n*-dimensional* C_v *for some cone* C *and vector* $v \in \mathbb{R}^n$, *then the bounded algebra* $B(K)$ *is trivial, i.e.,* $B(K) = \mathbb{R}$.

Proof:

If K contains an n -dimensional C_v then there exists an affine transformation $\Phi^* : \mathbb{R}^n \to \mathbb{R}^n, y \mapsto Ay + y_0$, where A is a real matrix of order $n \times n$, induced by the $\mathbb{R}$-algebra automorphism Φ of $\mathbb{R}[y]$ defined by $f(y) \mapsto f(Ay + y_0)$. The mapping Φ^* transforms K onto a region containing the first orthant $y_1 \geq 0, y_2 \geq 0, ..., y_n \geq 0$. Therefore, to prove the lemma, we need to show that $B(K) = \mathbb{R}$ in the case that K contains the first orthant.

Let $f(y) \in \mathbb{R}[y]$ be a polynomial of degree d such that $f(y)$ is bounded on K. We can write

$$f(y) = \sum_{i=0}^{d} f_i(y_1, y_2, ..., y_n), \tag{8}$$

where f_i is a homogeneous polynomial of degree i for every $i = 0,1,2,...,d$. Assume that $d > 0$. Since $f_d \neq 0$, there is a point $a = (a_1, a_2, ..., a_n)$ in the first orthant such that $f_d(a) \neq 0$ (such a point a exists because the dimension of the first orthant is n while that of the hypersurface $f_d^{-1}(0)$ is $n-1$). Take a parameter curve $y(t) = at; t \in (0, +\infty)$. Then $y(t)$ belongs to the first orthant, hence it is contained in K. In addition, the degree of the single variable polynomial $f\left(y(t)\right) = f_d(a)t^d + ... + f_0(a)$ is $d > 0$ so $f\left(y(t)\right)$ is unbounded on $(0, +\infty)$. This contradicts with the hypothesis of $f(y)$. Therefore, $d = 0$ and $B(K) = \mathbb{R}$.

Proof of Proposition 2.1:

- Case 1: If $\lim_{x \to +\infty} \left(g_2(x) - g_1(x)\right) = +\infty$ then there exist $a > 0$ and $k \in \mathbb{N}$ such that

$$g_2(x) - g_1(x) \sim ax^k \text{ as } x \to +\infty. \tag{9}$$

Changing of variables $z = y - g_1(x)$, we have that (x, y) belongs to $M(g_1, g_2)$ if and only if (x, z) belongs to $M := \left\{(x,z) \in \mathbb{R}^2 : 0 \leq z \leq g_2(x) - g_1(x)\right\}$. By the property (9), there exists x_1 such that $\left\{(x,z) \in \mathbb{R}^2 : 0 \leq z \leq \frac{a}{2}x, x \geq x_1\right\} \subset M$. So M contains the 2-dimensional translation C_v of the cone $C = cone\left((1,0); (1, \frac{a}{2})\right)$ by $v = (x_1, 0)$. From Lemma 2.2, the bounded algebra $B(M)$ is trivial. Therefore, the algebra of polynomials which are bounded on $M(g_1, g_2)$ is also $\mathbb{R}$.

- Case 2: If $\lim_{x\to-\infty}\left(g_2(x)-g_1(x)\right)=+\infty$ then by changing of variable $x=-t,$ we have

$$\lim_{t\to+\infty}\left(g_2(-t)-g_1(-t)\right)=\lim_{x\to-\infty}\left(g_2(x)-g_1(x)\right)=+\infty.$$

Put $\overline{M}:=\left\{(t,y)\in\mathbb{R}^2: g_1(-t)\le y\le g_2(-t)\right\}$. Then (x,y) belongs to $M(g_1,g_2)$ if and only if (t,y) belongs to $\overline{M}$. According to Case 1, we obtain $B(\overline{M})=\mathbb{R}$. Hence, the bounded algebra $B\left(M(g_1,g_2)\right)$ is also $\mathbb{R}$.

Remark 2.2: In view of the proof of Proposition 2.1, if we replace the strip $M(g_1,g_2)$ by the corresponding half strip:

$$M(g_1,g_2):=M(g_1,g_2)\cap\left\{(x,y)\in\mathbb{R}^2: x\ge x_0\right\}, \tag{10}$$

then the algebra of polynomials bounded on the half strip is the same as the one on the strip if $g_1<g_2$ at positive infinity ($\lim_{x\to+\infty}\left(g_2(x)-g_1(x)\right)=+\infty$). We use the same argument as Case 1 in the proof of Proposition 2.1. Performing change of variables $z=y-g_1(x)$, we find that (x,y) belongs to the half strip $M(g_1,g_2)$ if and only if (x,z) belongs to $M':=\left\{(x,z)\in\mathbb{R}^2: 0\le z\le g_2(x)-g_1(x), x\ge x_0\right\}$. By the property (9), there exists $x_1\ge x_0$ such that M' contains 2-dimensional translation C_v of the cone $C=cone\left((1,0);(1,\frac{a}{2})\right)$ by $v=(x_1,0)$. From Lemma 2.2, the bounded algebra $B(M')$ is trivial. Hence $B\Big(M(g_1,g_2)\Big)$ is also $\mathbb{R}$.

From Theorem 2.1 and Proposition 2.1, we get the following corollaries.

Corollary 2.2. *Let* $g_1(x), g_2(x)$ *be single variable polynomials and* $M(g_1,g_2)$ *defined in eq. (6). Then the following statements hold.*

1. *If* $g_2(x)-g_1(x)$ *is equal to a positive constant* c *then* $B\left(M(g_1,g_2)\right)=\mathbb{R}[y-g_1(x)]$.
2. *If* $g_1=g_2$ *then* $B\left(M(g_1,g_2)\right)=\left(y-g_1(x)\right)\mathbb{R}[x,y]+\mathbb{R}$.

Proof:

1. The first statement follows directly from Theorem 2.1 by $M(g_1,g_2)=M_c(g_1)$.
2. If $g_1=g_2$ then $M\left(g_1,g_2\right)=\left\{(x,y)\in\mathbb{R}^2: y=g_1(x)\right\}$.

It is clear that, $\left(y-g_1(x)\right)\mathbb{R}[x,y]+\mathbb{R}$ is a subset of $B\left(M(g_1,g_2)\right)$. Conversely, let $f(x,y)$ be an element of $B\left(M(g_1,g_2)\right)$. Then we can write

$$f(x,y)=f_1(x,y)\left(y-g_1(x)\right)+f_0(x),$$

where $f_1(x,y) \in \mathbb{R}[x,y]$; $f_0(x) \in \mathbb{R}[x]$. Thus, $f(x,y)$ is bounded on $M(g_1,g_2)$ if and only if $f_0(x)$ is bounded on $\mathbb{R}$. Hence, there is a constant $a \in \mathbb{R}$ such that $f_0(x) = a, \forall x \in \mathbb{R}$. This means that $f(x,y)$ belongs to $\left(y - g_1(x)\right)\mathbb{R}[x,y] + \mathbb{R}$.

Since $B(K_1 \bigcup K_2) = B(K_1) \bigcap B(K_2)$ with $K_1, K_2 \subset \mathbb{R}^n$, Theorem 2.1 and Proposition 2.1 can be stated for a finite union of generalized strips as follows.

Corollary 2.3. *Let $k \in \mathbb{N}$ and $g_1(x), g_2(x), ..., g_{2k}(x) \in \mathbb{R}[x]$ be distinct polynomials such that $\lim_{x\to+\infty}\left(g_{i+1}(x) - g_i(x)\right) > 0, \forall i = \overline{1, 2k-1}$. Let $K(g_1,...,g_{2k})$ be a subset of $\mathbb{R}^2$ defined by*

$$K(g_1,...,g_{2k}) := \left\{(x,y) \in \mathbb{R}^2 : \prod_{i=1}^{2k}\left(y - g_i(x)\right) \le 0\right\}. \tag{11}$$

Then the following statements hold.

1. *If there exists an index $i_0 \in \{1,2,...,2k-1\}$ such that $\lim_{x\to+\infty}\left(g_{i_0+1}(x) - g_{i_0}(x)\right) = +\infty$ then $B\left(K(g_1,...,g_{2k})\right)$ is trivial.*

2. *If $\lim_{x\to+\infty}\left(g_{i+1}(x) - g_i(x)\right)$ is a positive constant for al $i \in \{1,2,...,2k-1\}$ then $B\left(K(g_1,...,g_{2k})\right)$ is equal to $\mathbb{R}[y - g_1(x)]$.*

Proof:

1. By the assumption $\lim_{x\to+\infty}\left(g_{i+1}(x) - g_i(x)\right) > 0, \forall i = \overline{1, 2k-1}$, there exists $x_0 \in \mathbb{R}$ such that for all $x \ge x_0$, we have $g_1(x) < g_2(x) < ... < g_{2k}(x)$. Using the notation in Remark 2.2, we get $M(g_{2i-1}, g_{2i}) \subset K(g_1,...,g_{2k})$ for all $i = \overline{1,...,k}$. This deduces $\bigcup_{i=1}^{k} M(g_{2i-1}, g_{2i}) \subset K(g_1,...,g_{2k})$. Therefore,

$$B(K(g_1,...,g_{2k})) \subset B\left(\bigcup_{i=1}^{k} M(g_{2i-1}, g_{2i})\right) = \bigcap_{i=1}^{k} B\left(M(g_{2i-1}, g_{2i})\right). \tag{12}$$

Since $\mathbb{R} \subset B\left(K(g_1,...,g_{2k})\right)$, to prove the first statement we only need to show that $B\left(K(g_1,...,g_{2k})\right) \subset \mathbb{R}$.

- Case 1: If $i_0 = 2j-1$ for some $j \in \{1,2,...,k\}$ then $\lim_{x\to+\infty}\left(g_{2j}(x) - g_{2j-1}(x)\right) = +\infty$ by the assumption. So according to Remark 2.2, we get $B\left(M(g_{2j-1}, g_{2j})\right) = \mathbb{R}$. Hence, by the property (12), we deduce $B\left(K(g_1,...,g_{2k})\right) \subset \mathbb{R}$.
- Case 2: If $i_0 = 2j$ for some $j \in \{1,2,...,k\}$ then $\lim_{x\to+\infty}\left(g_{2j+1}(x) - g_{2j}(x)\right) = +\infty$.

If $\lim_{x\to+\infty}\left(g_{2j}(x)-g_{2j-1}(x)\right)=+\infty$ or $\lim_{x\to+\infty}\left(g_{2j+2}(x)-g_{2j+1}(x)\right)=+\infty$ then from Case 1, we have $B\left(K(g_1,...,g_{2k})\right)=\mathbb{R}.$ Now, we assume

$$\lim_{x\to+\infty}\left(g_{2j}(x)-g_{2j-1}(x)\right)=c_j;\ \lim_{x\to+\infty}\left(g_{2j+2}(x)-g_{2j+1}(x)\right)=c_{j+1},$$

where c_j,c_{j+1} are positive constants. Since $g_i\in\mathbb{R}[x]$, for $i=2j-1,...,2j+2$, this implies

$$g_{2j}=g_{2j-1}+c_j,\ g_{2j+2}=g_{2j+1}+c_{j+1}. \tag{13}$$

By Remark 2.1, we obtain:

$$B\Big(M(g_{2j-1},g_{2j})\Big)=\mathbb{R}[y-g_{2j-1}(x)]=\mathbb{R}[y-g_{2j}(x)], \tag{14}$$

$$B\Big(M(g_{2j+1},g_{2j+2})\Big)=\mathbb{R}[y-g_{2j+1}(x)]. \tag{15}$$

We next prove that $B\Big(M(g_{2j-1},g_{2j})\Big)\cap B\Big(M(g_{2j+1},g_{2j+2})\Big)=\mathbb{R}$ by showing $\mathbb{R}[y-g_{2j}(x)]\cap\mathbb{R}[y-g_{2j+1}(x)]=\mathbb{R},$ see eq. (14) and eq. (15). Let $f(x,y)\in\mathbb{R}[y-g_{2j}(x)]\cap\mathbb{R}[y-g_{2j+1}(x)].$ Since $f(x,y)\in\mathbb{R}[y-g_{2j}(x)],$ we have the following representation

$$f(x,y)=\sum_{i=0}^{m}a_i\left(y-g_{2j}(x)\right)^i;\ a_i\in\mathbb{R};\ i=\overline{1,m}.$$

We can rewrite $f(x,y)$ in the form

$$\begin{aligned}f(x,y)&=\sum_{i=0}^{m}a_i\left[\left(y-g_{2j+1}(x)\right)+\left(g_{2j+1}(x)-g_{2j}(x)\right)\right]^i\\&=\sum_{i=1}^{m}h_i(x)\left(y-g_{2j+1}(x)\right)^i+\sum_{i=0}^{m}a_i\left(g_{2j+1}(x)-g_{2j}(x)\right)^i;\ h_i(x)\in\mathbb{R}[x].\end{aligned} \tag{16}$$

By $f(x,y)\in\mathbb{R}[y-g_{2j+1}(x)]$ and eq. (16) we get

$$\begin{cases}h_i(x)\in\mathbb{R},i=\overline{1,m}\\ q(x):=\displaystyle\sum_{i=0}^{m}a_i\left(g_{2j+1}(x)-g_{2j}(x)\right)^i\in\mathbb{R}.\end{cases}$$

Observe that the hypothesis $\lim_{x\to+\infty}\left(g_{2j+1}(x)-g_{2j}(x)\right)=+\infty$ implies $\deg\left(g_{2j+1}(x)-g_{2j}(x)\right)>0.$ So if $m>0$ then $q(x)\notin\mathbb{R}.$ Hence, $m=0$ or $f(x,y)=a_0\in\mathbb{R}.$ Therefore $B\Big(M(g_{2j-1},g_{2j})\Big)\cap B\Big(M(g_{2j+1},g_{2j+2})\Big)=\mathbb{R}.$ By the property (12) we conclude that $B\left(K(g_1,...,g_{2k})\right)\subset\mathbb{R}.$

2. Assume that $\lim_{x\to+\infty}\left(g_{i+1}(x)-g_i(x)\right)=c_i>0$ for all $i\in\{1,2,...,2k-1\}$. We have

$$g_{i+1}(x)=g_i(x)+c_i=g_1(x)+c_1+...+c_i,\forall x\in\mathbb{R},i=\overline{1,2k-1}. \tag{16}$$

Using the notation in Proposition 2.1 and Theorem 2.1, we have

$$K(g_1,...,g_{2k}) = \bigcup_{i=1}^{k} M(g_{2i-1}, g_{2i}) = \bigcup_{i=1}^{k} M_{c_{2i-1}}(g_{2i-1}). \tag{17}$$

Otherwise, by Corollary 2.2 and eq. (16), we get

$$B\left(M(g_{2i-1}, g_{2i})\right) = \mathbb{R}[y - g_{2i-1}(x)] = \mathbb{R}[y - g_1(x) - c_1 - c_2 - ... - c_{2i-2}] = \mathbb{R}[y - g_1(x)].$$

From eq. (17), we find

$$B\left(K(g_1,...,g_{2k})\right) = \bigcap_{i=1}^{k} B\left(M(g_{2i-1}, g_{2i})\right) = \mathbb{R}[y - g_1(x)].$$

4. CONCLUSION

In this paper, we have described the algebra of polynomials bounded on some strips. On the Euclidean space of dimension $n+1$, we show that the algebra bounded on the generalized strip $M_c(g_1, g_2, ..., g_n)$ is finitely generated by the polynomials $y - g_1(x), ..., y - g_n(x)$ provided that c is a positive vector of $\mathbb{R}^n$. On the plane $\mathbb{R}^2$, the algebra bounded on the strip $M(g_1, g_2)$ depends on the limit $\lim_{x\to\infty}(g_2(x) - g_1(x))$. The algebra is trivial if the limit is positive infinity; finitely generated by $y - g_1(x)$ if this limit is a positive constant and equal to $(y - g_1(x))\mathbb{R}[x, y] + \mathbb{R}$ if this limit is zero. In addition, we also gave a corollary of this algebra on the finite union of generalized strips.

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