Transport and Communications Science Journal

ANALYSIS OF PERTURBES FRICTIONAL CONTACT PROBLEMS

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ARTICLE INFO

TYPE: Research Article
Received: 30/10/2021
Revised: 28/12/2021
Accepted: 05/01/2022
Published online: 15/01/2022
https://doi.org/10.47869/tcsj.73.1.6
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Abstract. Generally, friction/perturbation compensation is an important issue in high-precision motion control applications. In particular, when the system under control undergoes low-speed or reversal motions, the friction force and external perturbations will degrade motion accuracy. In this paper, we study analysis of perturbed frictional contact problems. Its shows how homotopy perturbation techniques and projection can be integrated in control-based approach to reanalyze the perturbed frictional contact problems. Thus, the perturbed non linear problem is decomposed into perturbed linear problems dedicated to each component in contact. Each solution of perturbed linear problems is approximated. A numerical application is performed to verify the efficiency and the robustness of the proposed method. The proposed method has been developed to be compatible within a context of multiple sampling (such as parametric analysis or design of experiments). The proposal relies on a control based method currently used in automation domain. A Fuzzy Logic Controller (FLC) is designed to link the normal gaps identified between the bodies and the normal contact pressures applied at the interface. Finally, a design of experiments is proposed to quantify the effects of input perturbations on output mechanical data.

Keywords: Perturbe, frictional contact, Fuzzy Logic Controller, Homotopy Perturbation and Projection .

1. INTRODUCTION

During the manufacturing of mechanical structures, it is not uncommon to observe some uncertainties resulting in product variability either on material properties (Young modulus, strain-stress law), on geometric characteristics (gaps, fillets and small geometries) and on interface and boundary conditions. These observed variabilities necessarily affect the static and dynamic behaviours of structures and more generally the component life and the global efficiency of the system. To significantly characterize experimental behaviour, it is so necessary to consider a family of components rather than only one specimen that can generate significant financial costs. Numerically, to take into account these uncertainties and tend to reliable and robust designs, a current industrial trend involves making multiple numerical simulations by performing sensitivity analyses, designs of experiments non-deterministic studies or even reliable and robust optimizations.

There are studies that have allowed to develop a new way for solving a frictional contact problem by considering a Fuzzy Logic Control (FLCs) approach [1-6]. Practically, the normal and tangential (sticking) contact loads are iteratively calculated as a function of observed gaps thanks to two FLCs implemented in parallel. To update the positions of contact nodes at each iteration, reduced linear contact problems are solved. The reduction is performed by static modes of each structure in contact [7,8]. As highlighted, a current trend is to multiply the number of samplings and used advanced strategies (DOE, sensitivity analyses, non-deterministic analyses or robust optimization) to tend to more robust and reliable designs [9,10]. In this context of multiple samplings, if perturbations are introduced on input parameters of the problem, the finite element matrices of the problem and so the equilibrium position and the contact data are necessarily modified. To maintain the computational time of these advanced approaches, it is necessary to integrate reduced order or reanalysis techniques [11,12].

In this paper, as the contact problem has been rewritten as a function of static modes, we propose to investigate the reanalysis of these data as a function of the introduced perturbations. Thus, these modified displacements are approximated by considering homotopy perturbation method [11], which allows to develop the solutions of the nonlinear problem as a series expansion (in the present, the introduced perturbation is seen as a nonlinearity). The method employs a homotopy transform to generate a convergent series solution of mathematical problem [13]. Finally, the obtained high order displacements will be respectively introduced in a projection matrix to reduce the size of each linear problems to be solved and built the contact projection matrix

Thus, first we summarizes the main equations of a nominal contact problem in a finite element context, the method used to solve the nonlinear problem and the definition of a modified reduced contact problem. In a context of multiple modifications, we presents the reanalysis of the modified projection matrix and more particularly the reanalysis of modified static modes by homotopy perturbation and projection techniques. To verify the efficiency of proposed method, a complete numerical application is presented. The results obtained for different classes of perturbations are confronted to those obtained using industrial code Abaqus in terms of precision and computational time. Finally, a Design of Experiments (DOE) is proposed to quantify the effects of input perturbations on output mechanical data.

2. PERTUBATIONS IN A STATIC FINITE ELEMENT CONTACT PROBLEM

In a finite element context, the nominal equilibrium equation, which governs the static problem [13], is written as follows:

$$K^{(0)}U^{(0)} + F_n^{(0)}(U^{(0)}) + F_t^{(0)}(U^{(0)}) = F_{\mathbf{ext}}^{(0)} \tag{1}$$

where the upper script "(0)" defines the nominal or initial configuration and $K^{(0)}$), $U^{(0)}$), $F_{\mathbf{ext}}^{(0)}$, $F_n^{(0)}(U^{(0)})$ and $F_t^{(0)}(U^{(0)})$ respectively represent the nominal stiffness matrix, the nominal nodal displacement vector, the nominal external load vector and the nominal normal and tangential contact loads. The size of the finite element problem is $\mathbf{n}_{\text{dof}}$. The problem studied is non-linear because the contact loads $F_n^{(0)}(U^{(0)})$ and $F_t^{(0)}(U^{(0)})$ depend directly on the unknown displacement $U^{(0)}$ of all bodies in contact. These two contact load vectors will be assembled by considering the local normal and tangential contact loads detected for each active contact pair of the finite element model.

The Fuzzy Logic Controller for Contact (FL2C) method is an iterative method, which decomposes the nonlinear contact problem into a series of linear problems where the contact loads are considered as external loads. Thus, a correction of contact loads and displacements is realized at each step by using Fuzzy Logic Controllers (FLC) to link the observed gaps with contact loads [14]. The equilibrium equation (1) is rewritten as follows:

$$K^{(0)}U^{(0,k+1)} = F_{\mathbf{ext}}^{(0)} + F_n^{(0,k+1)} + F_t^{(0,k+1)} \tag{2}$$

where $F_n^{(0,k+1)}$ and $F_t^{(0,k+1)}$ represent the normal and tangential contact loads at the k$^{\text{th}}$ iteration relative to the observed normal and tangential gaps.Then, each contact load is calculated as follows:

$$\begin{aligned}
F_n^{(0,k+1)} &= F_n^{(0,k)} + \Delta F_n^{(0,k+1)} \\
\Delta F_n^{(0,k+1)} &= \gamma \Delta F_n^{(0,k)} \\
F_t^{(0,k+1)} &= F_t^{(0,k)} + \Delta F_t^{(0,k+1)} \\
\Delta F_t^{(0,k+1)} &= \beta \Delta F_t^{(0,k)}
\end{aligned} \tag{3}$$

where $\boldsymbol{\gamma}$ and $\boldsymbol{\beta}$ are respectively normal and tangential coefficients associated with the increment of normal and tangential loads and determined with FL2C

3. REANALYSIS OF THE MODIFIED PROJECTION MATRIX

First, the modified stiffness matrices $\mathbf{K}^{(\mathbf{m})}$ of each component in contact can be decomposed as a function of the nominal stiffness matrices $\mathbf{K}^{(\mathbf{0})}$ and perturbed parts $\Delta\mathbf{K}$ [14], such as

$$K^{(m)} = K^{(0)} + \varepsilon \Delta K \tag{4}$$

Secondly, the modified vectors of the modified projection matrix, namely modified static modes $\mathbf{\Gamma}^{(\mathbf{m})}$ can be developed as a series of nominal vectors and different perturbed vectors such as:

$$\Gamma^{(m)} = \Gamma^{(0)} + \varepsilon^{1}\Gamma^{(1)} + ... + \varepsilon^{n1}\Gamma^{(n_1)} \quad (5)$$

where n_1 is the order of truncation of the development.

The homotopy perturbation technique allows a non-linear problem to be expressed as a set of linear problems. In the present case, the perturbation included in the problem is seen as the non-linearity and an additional unknown parameter $\boldsymbol{\varepsilon}$ is introduced to highlight this. Thus, the parameter $\boldsymbol{\varepsilon}$ is positioned in equation 4 alongside the perturbation or high order quantities. At the end of the formulation, parameter $\boldsymbol{\varepsilon}$ will be established to practically evaluate the perturbed solutions. The calculation of modified static modes and modified eigenmodes are detailed in the next subsection. The linear system of equations associated with the modified static modes is defined by

$$K^{(m)}\Gamma^{(m)} = F_I \quad (6)$$

where F_I is a vector of unitary load

Introducing equation 4 and 5 into equation 6, one obtains:

$$(K^{(0)} + \varepsilon\Delta K)(\Gamma^{(0)} + \varepsilon^{1}\Gamma^{(1)} + ... + \varepsilon^{n1}\Gamma^{(n1)}) = F_I \quad (7)$$

The nominal solution $\boldsymbol{\Gamma}^{(0)}$ is calculated using a standard algorithm after decomposing the nominal stiffness matrix $\mathbf{K}^{(0)}$.

$$K^{(0)}\Gamma^{(0)} = F_I \quad (8)$$

The perturbed vector $\boldsymbol{\Gamma}^{(i)}$ is calculated by identifying different order terms of parameter $\boldsymbol{\varepsilon}$

$$K^{(0)}\Gamma^{(i)} = -\Delta K\Gamma^{(i-1)} \quad \text{with i} = 1....n_1 \quad (9)$$

All calculated perturbed vectors $\boldsymbol{\Gamma}^{(i)}$ depend on the same matrix, $\mathbf{K}^{(0)}$, which will be decomposed only once during the process, whatever the number of perturbed problems. To ensure the quality of the approximation $\boldsymbol{\Gamma}^{(m)}$, a projection of the equilibrium onto a subspace spanned by the columns of a rectangular projection basis $T_\Gamma^{(m)}$ is considered. The projection basic $T_\Gamma^{(m)}$ is built by using solution $\boldsymbol{\Gamma}^{(0)}$ of the initial problem and perturbed solution $\boldsymbol{\Gamma}^{(i)}$, as written:

$$T_\Gamma^{(m)} = \left[\Gamma^{(0)}\Gamma^{(1)}....\Gamma^{(i)}...\Gamma^{n1}\right] \quad (10)$$

This projection basis, whose size is $\left[n_{dof} \times (n_{sta} \times (n_1 - 1))\right]$ is orthonormalized. The reduced modified static problem is thus defined as follows:

$$T_\Gamma^{(m)^T} K^{(m)} T_\Gamma^{(m)} \Gamma_R^{(m)} = T_\Gamma^{(m)^T} F_I \quad (11)$$

$$\Gamma^{(m)} = T_\Gamma^{(m)} \Gamma_R^{(m)} \quad (12)$$

Finally, the results $\boldsymbol{\Gamma}^{(m)}$ of equation 12 are used to update the projection matrix $\mathbf{T}^{(m)}$. The reanalysis method, based on Homotopy Perturbation and Projection, is named HPP method in the rest of the paper. The next section presents the coupling of the reanalysis and the contact methods.

4. PROPOSED HPP-FL2C METHOD

The proposed method, used to reanalyse a perturbed frictional contact problem, relies on the coupling of the HPP method and on the FL2C method. To facilitate understanding, a flowchart, summarizing the different steps and the principal equations, is presented in figure 1

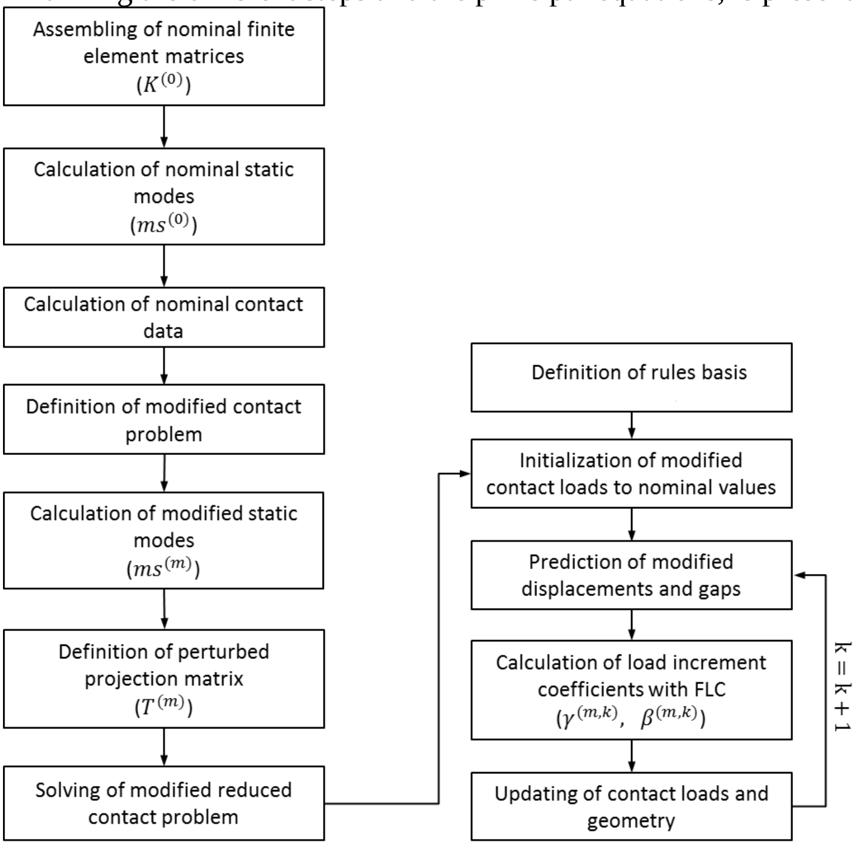

Figure 1. Flowchart of the proposed HPP-FL2C.

First, when the nominal finite element matrices of each structure of the system have been assembled, it is necessary to calculate the nominal static modes of each structure with only their own boundary conditions. These evaluations are performed using classical methods. The nominal solutions of the contact problem can be solved using, for example, the FL2C method. Secondly, for each modified problem, the modified projection matrix is built with a reanalysis method. Thirdly, the reduced modified problem is built by considering a projection of equilibrium equation on the modified projection matrix. Finally, the reduced contact problem is iteratively solved using the FL2C method. In summary, to validate the proposed developments step by step, four numerical methods will be considered in the following section dedicated to numerical application, and will be labelled as follows: Abaqus for calculations performed on Abaqus software (node to surface algorithm and a minimal increment fixed at $1{:}10^{-5}$ to guarantee the convergence). Full-FL2C for calculations performed using the control approach for the contact problem with full finite element matrices size. Proj-FL2C for calculations performed using the control approach for the reduced contact problem. HPP-FL2C for calculations performed using the control approach and reanalysis of the projection matrix.

5. NUMERICAL APPLICATIONS FOR FRICTIONAL CONTACT PROBLEM WITH PERTUBATIONS

5.1. Decription of the numerical model

A static frictional contact problem of two 2D semicircular steel rings is considered in figure 2. Each ring is defined by an outer radius of 100 mm and an inner radius of 95 mm. The initial position of the upper ring's center is fixed at 140 mm to the left and 190 mm up from the center of the lower ring. The finite element discretization for each ring includes 640 quadrinodal plane stress elements and 729 nodes per ring, corresponding to 8 divisions of the radius and 80 sectors.

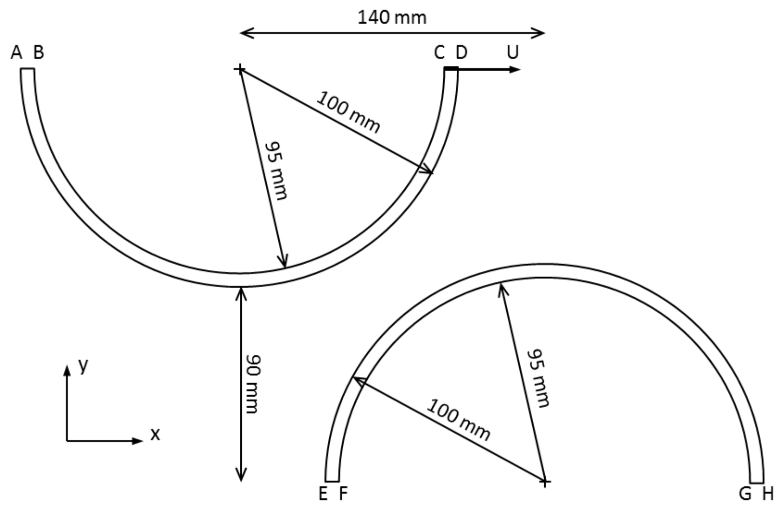

Figure 2. Description of the system studied composed of two elastic rings in contact.

The elastic material properties of the two rings are fixed at 210 GPa for Young's moduli and 0.3 for Poisson's ratios. Fixed boundary conditions are applied to the GH segment along the x and the y-axes, AB, CD and EF segments along the y-axis only. The CD segment is controlled in displacement along the x-axis up to 240 mm. Twenty-four steps are performed for this problem, so an horizontal displacement of 10 mm is applied to CD at each step. At the interface, a friction coefficient is set at 0.5. In the following analyses, the evolution of the sum of normal and tangential contact loads as well as the x-axis displacements of nodes A and E will be systematically studied for the different steps.

5.2. Nominal contact problem validation

The nominal results obtained with the Full-FL2C method are compared to those calculated with Abaqus. Figure 3 first presents the evolution of the x-axis displacements of nodes A, E.

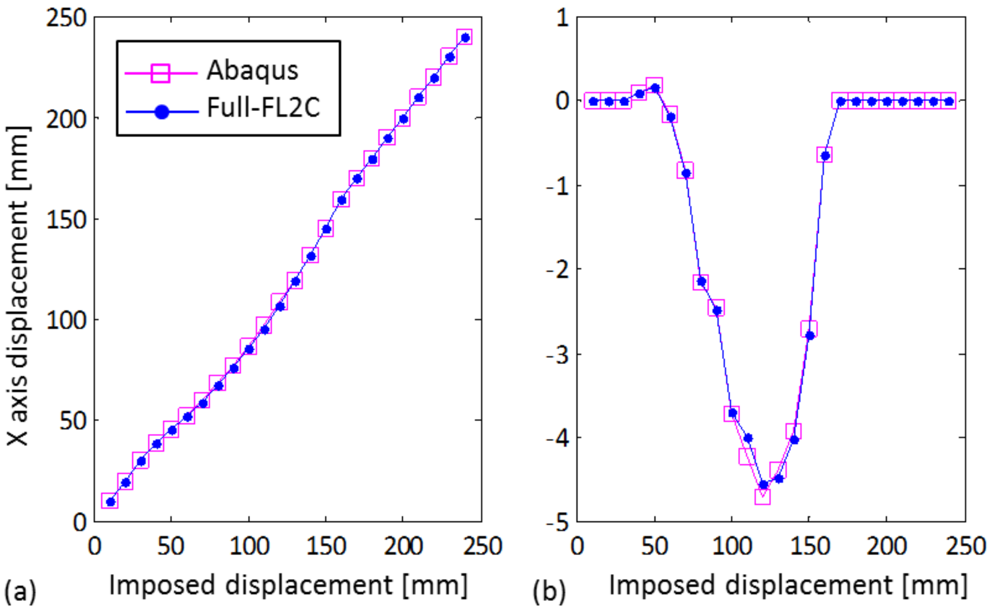

Figure 3. Comparison of x-axis displacement between Abaqus and Full-FL2C. (a) Node A; (b) node E.

The global evolution of displacements calculated with the Full-FL2C method is very close to that obtained with Abaqus. The maximum error is inferior to 1.5% and is found for node E around an imposed displacement of 110 mm. On the contrary, for lower and higher imposed displacements, errors are much lower as they do not exceed 1%. Besides, the maximal difference for the displacement following the x-axis of node A is in the order of 10^{-4}m for imposed displacement at 110 mm.

Now, the evolution of the sums of normal and tangential contact loads as a function of imposed displacements in figure 4 is assumed.

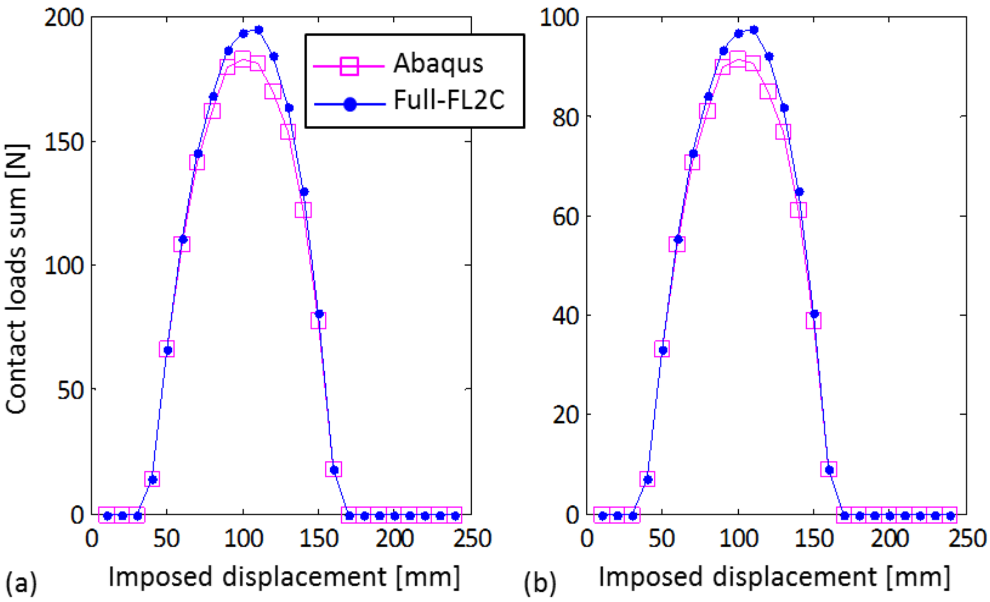

Figure 4. Comparison of the sum of contact loads between Abaqus and Full-FL2C: (a) Surm of normal contact loads; (b) Tangential contact loads

The results obtained are rather interesting because they are globally close to those given by the Abaqus solutions. The maximal errors for both normal and tangential loads are inferior to 8% and are detected only for imposed displacements around 110mm and 120mm. However, these errors have a low effect on the global kinematic behavior concerning the x-axis displacements of nodes A and E previously presented. In the present case, the mean normal gap for the Full-FL2C method after equilibrium is equal to 1µm. These differences are mainly linked to the distribution of contact loads between the two algorithms.

5.3. Nominal reduced contact problem validation

This section is dedicated to validation of the nominal reduced problem and the use of a projection matrix to decrease the size of the contact problem and therefore the computational time. A discussion is offered on the number of modal and static modes which must be kept to achieve a good approximation of mechanical output data. Figures 5 and 6 present the evolution of x-axis displacements and the sum of contact loads for Full-FL2C and Proj-FL2C as a function of the number of modes in the projection matrix. Three cases are studied:10 eigenmodes for each elastic ring and no static modes (case 1); 10 eigenmodes for each elastic ring and 112 static modes (case 2); 112 static modes (case 3).

One can observe that it is necessary to consider the static modes of each ring calculated along both the x-axis and the y-axis at the contact interface. The errors are very close to those obtained with the Full-FL2C method. The maximal error of contact loads (normal and tangential) is in the order of 10^{-3}%, while the maximal difference for the x-axis displacement of node A and E is in the order of 10^{-4} mm. In this case, considering modal basis does not improve the results to any greater degree. On the other hand, a lack of static modes clearly deteriorates the solutions

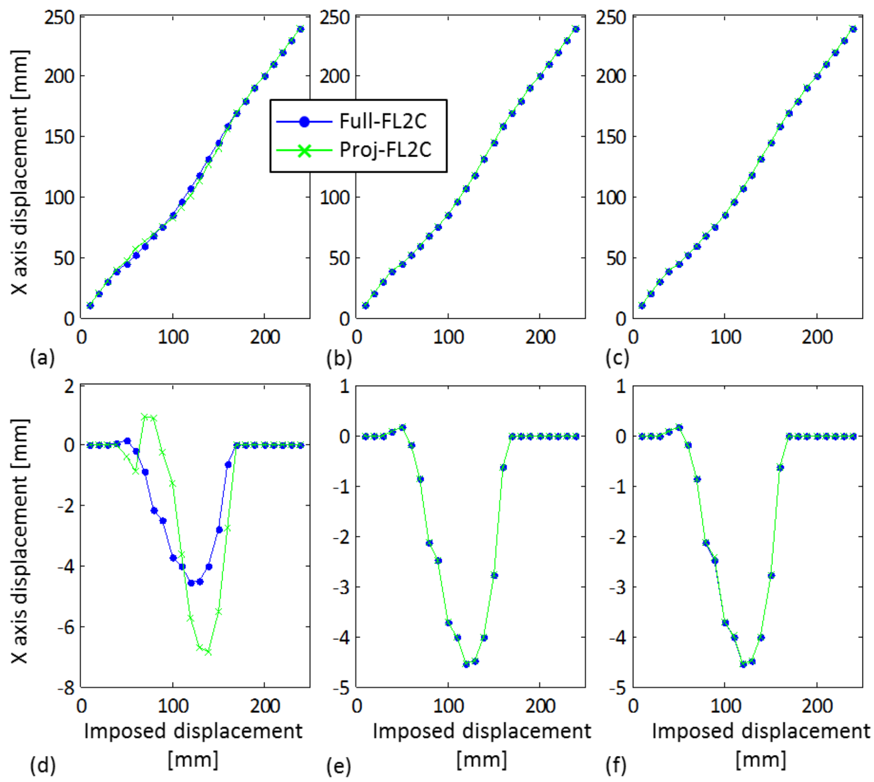

Figure 5. Comparison of x-axis displacement between Full-FL2C and Proj-FL2C for different projection matrices. (a;b;c) Node A; (d;e;f) node E; (a;d) case 1; (b;e) case 2; (c;f) case 3

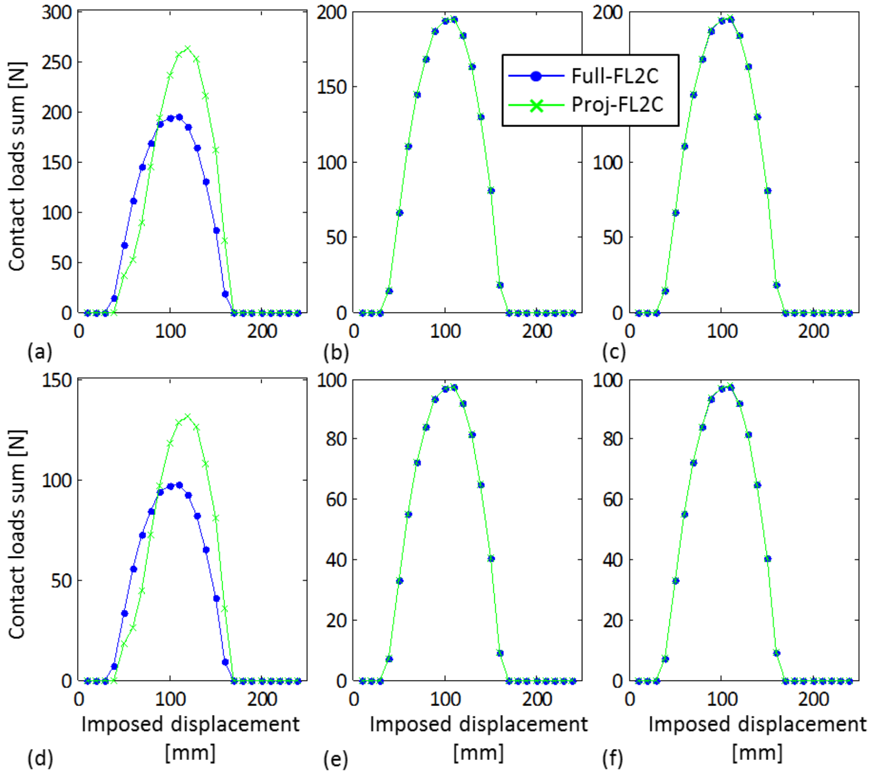

Figure 6. Comparison of the sum of contact loads between Full-FL2C and Proj-FL2C for different projection matrices. (a;b;c) Sum of normal contact loads; (d;e;f) sum oof tangential contact loads;(a;d) case 1; (b;e) case 2; (c;f) case 3

5.4. Perturbed reduced contact problem

The first order HPP-FL2C method is considered to reanalyze the projection matrix of the perturbed reduced contact problem. The two kinds of perturbations will be tested as previously, namely Young's moduli, respectively equal to 215 GPa and 205 GPa for the lower and upper rings, with an inner radius of 96 mm for the lower ring and 94 mm for the upper ring. The friction coefficient is fixed at its nominal value, namely 0.5. Figures 7 and 8 present a comparison of results between Abaqus and HPP-FL2C by considering firstly, the x-axis displacements for node A and E and secondly, the sums of normal and tangential contact loads. The reanalyzed x-axis displacements calculated with the HPP-FL2C method are very closeto those obtained with Abaqus, whatever the perturbation. The mean errors are inferior to 1% for both nodes A and E. Moreover, the evolution of contact loads is globally well captured. For the totality of the simulation, the rate of error reaches a maximum of 10%. Thus, this study shows the efficiency of the HPP-FL2C method to calculate with precision perturbed mechanical output quantities such as displacements and contact loads.

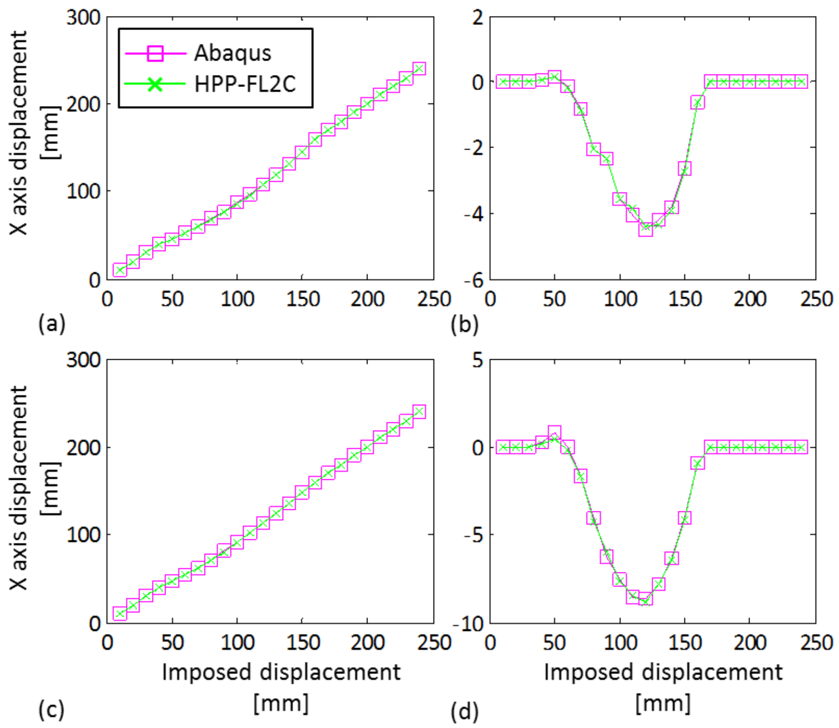

Figure 7. Comparison of x-axis displacements between Abaqus and HPP-FL2C for different categories of perturbations. (a;b) Material perturbation; (c;d) geometric perturbation; (a;c) node A; (b;d) node E

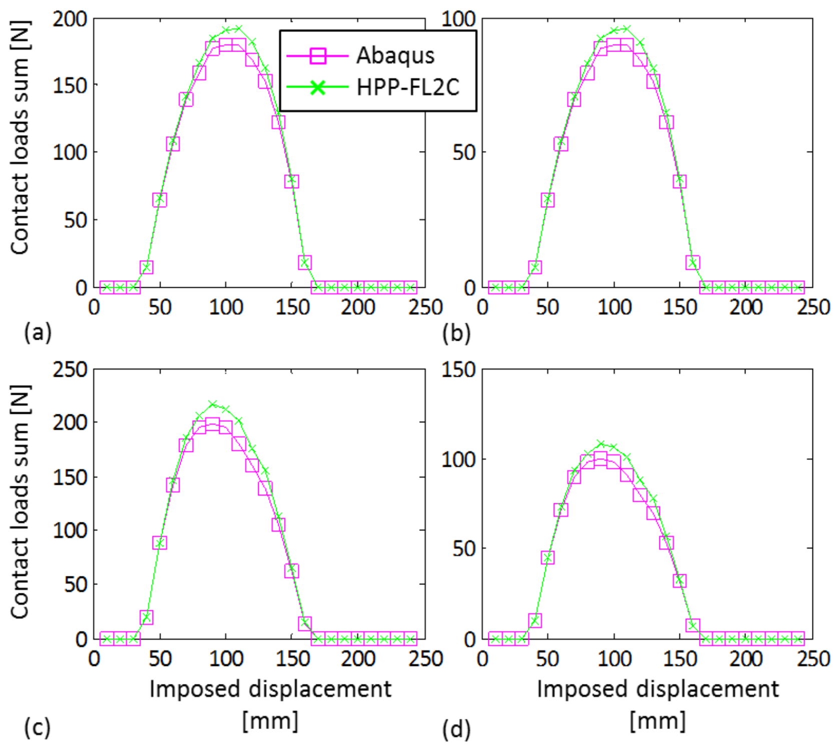

Figure 8. Comparison of contact loads between Abaqus and HPP-FL2C for different categories of perturbations. (a;b) Material perturbation; (c;d) geometric perturbation; (a;c) normal contact loads; (b;d) tangential contact loads.

Now, figure 9 presents the evolution of computational time for one reanalysis as a function of the finite element model size. Three methods, namely Abaqus, Full-FL2C and HPP-FL2C, are considered for this comparison. The number of degrees of freedom (dof) of the finite element model is successively fixed at 120, 400, 1400, 5500 and 21000 dofs for each ring.

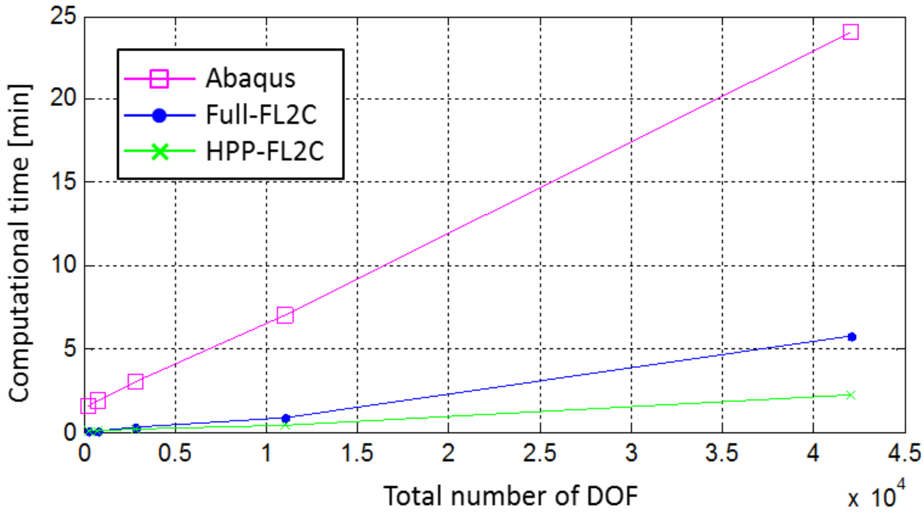

Figure 9. Comparison of computational time of Abaqus, Full-FL2C and HPP-FL2C methods as a function of finite element model size

The computational time for Full-FL2C and HPP-FL2C is always less than Abaqus, regardless of the model size. When the model size increases, the computational gain between Abaqus and HPP-FL2C gradually increases. For the largest model, the computational gain factor between Abaqus and Full-FL2C can reach 75% and almost 50% between Full-FL2C and HPP-FL2C. This section highlights the advantage of coupling a reanalysis technique and a control method to solve a perturbed contact problem.

6. PARAMETRIC ANALYSIS BY DOE (DESIGN OF EXPERIMENTS)

Considering the previous observations in terms of precision and computational time to reanalyze a perturbed contact problem, the last test examines the use of the HPP-FL2C method in a DOE. A full factorial DOE considering both an interface parameter (friction coefficient), a material parameter (Young's modulus of each ring) and geometric parameter (inner radius of each ring). The values of each category of parameters can vary respectively between 0.4 to 0.6 (±20%), 205 GPa to 215 GPa (±2.4%) and 94 mm to 96 mm (±1.05%). Figures 10 and 11 present the effects of input perturbations on x-axis displacements and contact loads calculated with the Abaqus and HPP-FL2C methods. First, the minimal and maximal evolutions of behaviours of output data are very close between the Abaqus and HPP-FL2C methods, which again confirms the precision of the proposed HPP method. We can note the significant effect of input perturbation on the x-axis displacement for node E. In fact, an output variation of ±90% is observed for node E against ±5% for node A. This study shows that moderate input perturbations can generate large variations in the output. The same significant evolution of behaviour is observed for the contact loads. Finally, in terms of computational time, this study was performed in approximately twelve hours using Abaqus compared with one hour using the HPP-FL2C method with Matlab, representing a computational gain factor of 12.

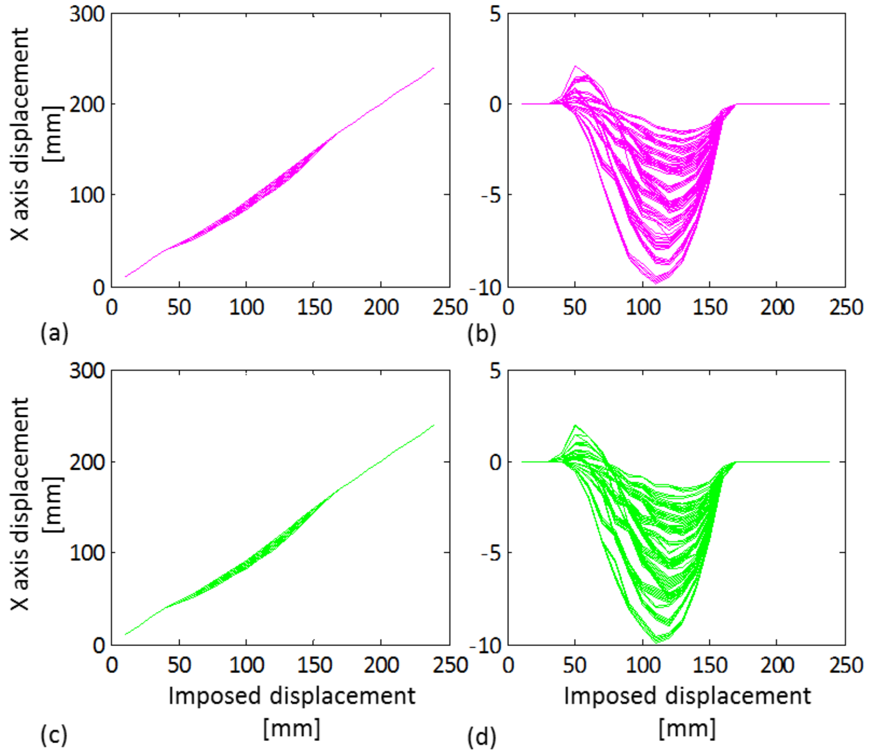

Figure 10. Comparison of DOE results obtained with Abaqus and HPP-FL2C for displacement, (a;b)Abaqus; (c;d) HPP-FL2C(a;c) x-axis displacement of node A; (b;d) x-axis displacement of node E.

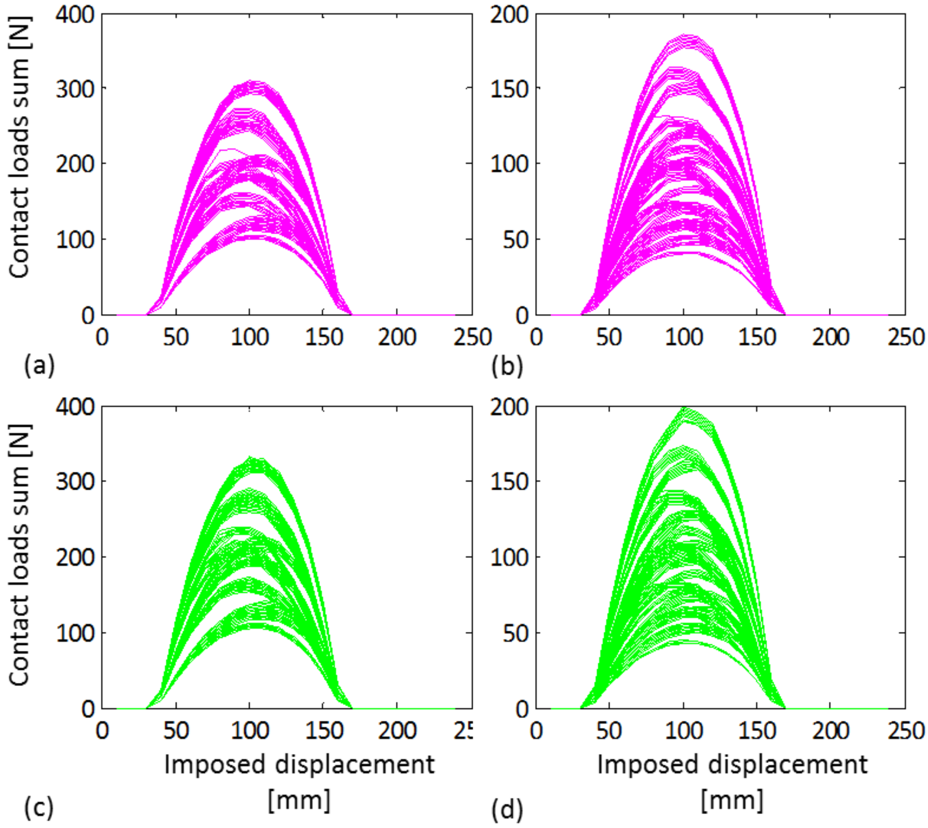

Figure 11. Comparison of DOE results obtained with Abaqus and HPP-FL2C of contact loads. (a;b) Abaqus; (c;d) HPP-FL2C; (a;c) normal contact loads; (b;d) tangential contact loads

7. CONCLUSION

This paper has presented a numerical strategy to reanalyze a modified static frictional contact problem. This one relies on the coupling of the FL2C method useful to the solving of the contact problem and the HPP method used for the reanalysis of coupling contact projection matrix. Indeed, in the context of multiple samplings, the input parameters are perturbed and affect the finite element matrices and so the static modes (which are the key elements of the contact projection matrix). Thus, we have proposed to reanalyse the modified static modes with a homotopy development and a new projection step. The different steps of the HPP-FL2C method have been developed.

Different tests have been performed to evaluate both the precision and the performance of the proposed HPP-FL2C method. We have highlighted that HPP-FL2C is a convenient strategy when parametric study such as DOE is considered. Indeed, with a first order of truncation, it is possible to obtain a level of precision compatible with advanced direct simulations. Moreover, an important gain of computational time can be obtained whichever category of variable parameter is involved namely interface, material or topology. We have shown that moderate input perturbations can generate large variations on the output contact data and both local and global modification of behaviours. Now, considering HPP-FL2C method, it is now possible to investigate the reanalysis of the equilibrium position of frictional induced vibration problems

ACKNOWLEDGMENT

This research is funded by VAST under grant number VAST01.08/21-22

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