EXISTENCE AND UNIQUENESS OF SOLUTIONS FOR TWO-DIMENSIONAL FRACTIONAL NON-COLLIDING PARTICLE SYSTEMS

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Abstract. In this paper, we consider the stochastic evolution of two particles with electrostatic repulsion and restoring force which is modeled by a system of stochastic differential equations driven by fractional Brownian motion where the diffusion coefficients are constant. This is the simplest case for some classes of non-colliding particle systems such as Dyson Brownian motions, Brownian particles systems with nearest neighbour repulsion. We will prove that the equation has a unique non-colliding solution in path-wise sense.

Keywords: stochastic differential equation, fractional Brownian motion, non-colliding particle systems.

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1. INTRODUCTION

It is known that the systems of SDEs driven by standard Brownian motion describing positions of $d$ ordered particles evolving in $R$ has the form

$$dx_i(t) = \left\{ \sum_{j \neq i} \frac{y_{ij}}{x_i(t) - x_j(t)} + b_i(t,x(t)) \right\}dt + \sum_{j=1}^{m} \sigma_{ij}(x(t))dW_j(t), i = 1...d, \quad (1)$$

where $W = (W_1(t), W_2(t),..., W_m(t))$ is a $m$-dimensional standard Brownian. The system of SDEs (2) is a type of SDEs whose solution stays in a domain which has been studied by many
authors because of its important applications in physics, biology and finance [1]. In mathematical physics, the process $x(t)$ is used to model systems of non-colliding particles with electrostatic repulsion and restoring force. It contains Dyson Brownian Motions, Squared Bessel particle systems, Jacobi particle systems, non-colliding Brownian and Squared Bessel particles, potential-interacting Brownian particles and other particle systems crucial in mathematical physics and physical statistics [2, 3]. The existence and uniqueness of a strong non-colliding solution to such kind of systems have been intensively studied by many authors ([4, 5, 6, 7] and the references therein). But there are no results in the case of fractional non-colliding particles.

The main aim of this paper is to study the two-dimensional fractional non-colliding particle systems

$$
\begin{align*}
    dX_1(t) &= \left(\gamma X_1(t) + b_1(t, X(t))\right) dt + \sum_{j=1}^{m} \sigma_{ij} dB_j^H(t), \\
    dX_2(t) &= \left(\gamma X_2(t) + b_2(t, X(t))\right) dt + \sum_{j=1}^{m} \sigma_{ij} dB_j^H(t),
\end{align*}
$$

(2)

where $X(0) = (X_1(0), X_2(0)) \in \Delta_2 = \{x = (x_1, x_2)^T \in \mathbb{R}^2 : x_1 < x_2\}$ almost surely (a.s) and $B = (B^H(t), t \geq 0) = (B_1^H(t), B_2^H(t), ..., B_m^H(t))^T$ is an $m$-dimensional fractional Brownian motion with the Hurst parameter $H \in (\frac{1}{2}, 1)$ defined on a complete probability space $(\Omega, \mathcal{F}, P)$ with a filtration $\{\mathcal{F}_t, t \geq 0\}$ satisfying the usual conditions. We prove that equation (1) has a unique non-colliding solution in path-wise sense. To the best of my knowledge, this is the first paper to discuss the fractional non-colliding particle systems.

2. THE EXISTENCE AND UNIQUENESS OF THE SOLUTION

Fix $T > 0$ and we consider eq. (1) on the interval $[0, T]$. We suppose that the coefficients $b_i : [0; +\infty) \times \mathbb{R}^2 \to \mathbb{R}$ are measurable functions and there exist positive constants $L, C$ such that following conditions hold

(i) $X(0) \in \Delta_2$ almost surely.

(ii) $\gamma > 0$.

(iii) $b_i(t,x), i = 1, 2$ are globally Lipschitz continuous with respect to $x$, that is

$$
\sup_{i=1,2} \left| b_i(t,x) - b_i(t,y) \right| \leq L \|x - y\|,
$$

for all $x, y \in \mathbb{R}^2$ and $t \in [0,T]$.

(iv) $b_i(t,x), i = 1, 2$ are sub-linearly growth with respect to $x$, that is

$$
\sup_{i=1,2} \left| b_i(t,x) \right| \leq C(1+|x|),
$$

for all $x \in \mathbb{R}^2$ and $t \in [0,T]$.

(v) $b_1(t,x) < b_2(t,x)$ for all $x \in \mathbb{R}^2$ and $t \in [0,T]$. 

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Denote $a \lor b = \max\{a, b\}$ and $a \land b = \min\{a, b\}$. For each $n \in \mathbb{N}$, we consider the following fractional SDEs

$\begin{align*}
    dX_1^n(t) &= \left(\frac{\gamma}{X_i^n(t) - X_2^n(t)} \land \frac{1}{n} + b_1(t, X^n(t))\right) dt + \sum_{j=1}^m \sigma_{1j} dB_{j}^H(t), \\
    dX_2^n(t) &= \left(\frac{\gamma}{X_2^n(t) - X_1^n(t)} \lor \frac{1}{n} + b_2(t, X^n(t))\right) dt + \sum_{j=1}^m \sigma_{2j} dB_{j}^H(t),
\end{align*}$

(3)

where $X^n(0) = (X^n_i(0), X^n_2(0)) \in \Delta_2$. For each $n \in \mathbb{N}$ and $x = (x_1, x_2)$ we set

$f_1^n(t, x) = \frac{\gamma}{(x_1 - x_2) \land \frac{1}{n}} + b_1(t, x),$

$f_2^n(t, x) = \frac{\gamma}{(x_2 - x_1) \lor \frac{1}{n}} + b_2(t, x).$

**Lemma 2.1.** For each $T > 0$, eq. (3) has a unique solution on $[0, T]$.

**Proof:** Using the estimate $|a \lor c - b \lor c| \leq |a - b|, |a \land c - b \land c| \leq |a - b|$, it is straightforward to verify that

$|f_1^n(t, x) - f_1^n(t, y)| \leq (\sqrt{2} \gamma n^2 + L) |x - y|,$

for all $x = (x_1, x_2)$ and $t \in [0, T]$ and

$|f_1^n(t, x)| \leq n \gamma + C(1 + |x|).$

It means that coefficients of eq. (3) satisfy Lipschitz continuity and boundedness condition. Hence it follows from Theorem 2.1 in [8] that eq. (3) has a unique solution on the interval $[0, T]$.

We recall a result on the modulus of continuity of trajectories of fractional Brownian motion ([9])

**Lemma 2.2.** Let $B = \{B^H(t), t \geq 0\}$ be a fractional Brownian motion of Hurst parameter $H \in (0, 1)$. Then for every $0 < \varepsilon < H$ and $T > 0$, there exists an event $\Omega_{e,t}$ with $P(\Omega_{e,t}) = 1$, and a positive random variable $\eta_{e,t}$ such that $E(\eta_{e,t}^p) < \infty$ for all $p \in [1, \infty)$ and for all $s, t \in [0, T]$,

$|B^H(t, \omega) - B^H(s, \omega)| \leq \eta_{e,t}(\omega)|t - s|^{H - \varepsilon},$ for any $\omega \in \Omega_{e,t}$.
We denote
\[ \tau_n = \inf\{ t \in [0,T] : \left| X^n_2(t) - X^n_1(t) \right| \leq \frac{1}{n} \} \wedge T. \]

In order to prove that eq. (1) has a unique solution on \([0,T]\), we need the following lemma.

**Lemma 2.3.** The sequence \( \tau_n \) is non-decreasing, and for almost all \( \omega \in \Omega \), \( \tau_n(\omega) = T \) for \( n \) large enough.

**Proof.** Using the estimate \( -(a \wedge b) = -a \vee -b \), from eq. (3) we have
\[
d(X^n_2(t) - X^n_1(t)) = \left( \frac{2\gamma}{(X^n_2(t) - X^n_1(t))} \wedge \frac{1}{n} \right) \, dt + \sum_{j=1}^{m} (\sigma_{2j} - \sigma_{1j}) dB^H_j(t). \]

We set \( Y^n(t) = X^n_2(t) - X^n_1(t) \). Eq. (4) becomes
\[
d(Y^n(t)) = \left( \frac{2\gamma}{Y^n(t)} \wedge \frac{1}{n} \right) \, dt + \sum_{j=1}^{m} (\sigma_{2j} - \sigma_{1j}) dB^H_j(t). \quad (5) \]

Then \( Y^n(0) > 0 \) and \( \tau_n = \inf\{ t \in [0,T] : \left| Y^n(t) \right| \leq \frac{1}{n} \} \wedge T. \)

It follows from Lemma 2.2 that for any \( \varepsilon \in (0,H - \frac{1}{2}) \), there exist a finite random variable \( \eta_{\varepsilon,T} \) and an event \( \Omega_{\varepsilon,T} \in \mathcal{F} \) which do not depend on \( n \) such that \( P(\Omega_{\varepsilon,T}) = 1 \), and
\[
\left| \sum_{j=1}^{m} (\sigma_{2j} - \sigma_{1j}) (B^H_j(t, \omega) - B^H_j(s, \omega)) \right| \leq \eta_{\varepsilon,T}(\omega) \left| t - s \right|^{H-\epsilon}, \text{ for any } \omega \in \Omega_{\varepsilon,T} \text{ and } 0 \leq s < t \leq T. \quad (6) \]

We will adapt the contradiction method in [10]. Assume that for some \( \omega_0 \in \Omega_{\varepsilon,T}, \tau_n(\omega_0) < T \) for all \( n \in \mathbb{N} \). By virtue of the continuity of sample paths of \( Y^n \), it follows from the definition of \( \tau_n \) that \( Y^n(\tau_n(\omega_0), \omega_0) = \frac{1}{n} \) and \( Y^n(t, \omega_0) \geq \frac{1}{n} \) for all \( t \in [0, \tau_n(\omega_0)] \). Denote
\[ \kappa_n(\omega_0) = \sup\{ t \in [0, \tau_n(\omega_0)] : Y^n(t, \omega_0) \geq \frac{2}{n} \}. \]

We have
\[
\frac{1}{n} \leq Y^n(t, \omega_0) \leq \frac{2}{n}, \text{ for all } t \in [\kappa_n(\omega_0), \tau_n(\omega_0)]. \]

In order to simplify our notations, we will omit \( \omega_0 \) in brackets in further formulas. We have
\[
Y^n(\tau_n) - Y^n(\kappa_n) = -\frac{1}{n} \int_{\kappa_n}^{\tau_n} \left( \frac{2\gamma}{Y^n(s)} + b_2(s, X^n(s)) - b_1(s, X^n(s)) \right) \, ds + \sum_{j=1}^{m} (\sigma_{2j} - \sigma_{1j}) (B^H_j(\tau_n) - B^H_j(\kappa_n)). \]

This implies
\[
\left| \sum_{j=1}^{m} (\sigma_{2j} - \sigma_{y}) (B_j^H (\tau_n) - B_j^H (\kappa_n)) \right| = \left| \frac{1}{n} + \int_{\kappa_n}^{\tau_n} \left( \frac{2\gamma}{Y^n(s)} + b_2(s, X^n(s)) - b_1(s, X^n(s)) \right) ds \right|. \tag{7}
\]

Note that for all \( s \in [\kappa_n, \tau_n] \)
\[
\frac{2\gamma}{Y^n(s)} + b_2(s, X^n(s)) - b_1(s, X^n(s)) \geq 4n\gamma.
\]

Then for all \( n \geq n_0 = \frac{2}{Y^n(0)} \), it follows from eq. (7) that
\[
\left| \sum_{j=1}^{m} (\sigma_{2j} - \sigma_{y}) (B_j^H (\tau_n) - B_j^H (\kappa_n)) \right| \geq \frac{1}{n} + 4n\gamma (\tau_n - \kappa_n).
\]

This fact together with eq. (6) implies that
\[
\eta_{x,t} |\tau_n - \kappa_n|^{\mu - \varepsilon} \geq \frac{1}{n} + 4n\gamma (\tau_n - \kappa_n), \text{ for all } n \geq n_0 \tag{8}
\]

By following similar arguments in the proof of Theorem 2 in [10], we see that the inequality (8) fails for all \( n \) large enough. This contradiction completes the proof of the lemma.

We consider the process \( \{ X(t) = (X_1(t), X_2(t)) \}_{t \geq 0} \) which satisfies equation (1). Now, we set \( Y(t) = X_2(t) - X_1(t) \), then \( Y(t) \) satisfies the following equation
\[
d(Y(t)) = \left( \frac{2\gamma}{Y(t)} + b_2(t, X(t)) - b_1(t, X(t)) \right) dt + \sum_{j=1}^{m} (\sigma_{2j} - \sigma_{1j}) dB_j^H (t). \tag{9}
\]

**Lemma 2.4.** If eq. (1) has a solution then \( Y(t) = X_2(t) - X_1(t) > 0 \) for all \( t \in [0,T] \) almost surely.

**Proof.** We will also use the contradiction method. Assume that for some \( \alpha_0 \in \Omega \), \( \inf_{[0,T]} Y(t, \alpha_0) = 0 \). Denote \( \tau = \inf \{ t : Y(t, \alpha_0) = 0 \} \). For each \( n \geq 1 \) we denote \( \nu_n = \sup \{ t < \tau : Y(t, \alpha_0) = \frac{1}{n} \} \). Since \( Y \) has continuous sample paths, \( 0 < \nu_n < \tau \leq T \) and \( Y(t, \alpha_0) \in (0, \frac{1}{n}) \) for all \( t \in (\nu_n, \tau) \). We have
\[
Y(\tau) - Y(\nu_n) = -\frac{1}{n} = \int_{\nu_n}^{\tau} \left( \frac{2\gamma}{Y(s)} + b_2(s, X(s)) - b_1(s, X(s)) \right) ds + \sum_{j=1}^{m} (\sigma_{2j} - \sigma_{1j}) (B_j^H (\tau) - B_j^H (\nu_n)).
\]

Note that for all \( s \in [\nu_n, \tau] \)
\[
\frac{2\gamma}{Y(s)} + b_2(s, X(s)) - b_1(s, X(s)) \geq 2n\gamma.
\]

So we have
Again using the inequality (6), we have
\[ \eta_{\varepsilon,T} \left| \tau - \nu_n \right|^{\mu-\varepsilon} \geq \frac{1}{n} + 2n\gamma (\tau - \nu_n). \]  
(11)

Similar to the argument of Theorem 2 in [10] we see that the inequality (11) fails for all \( n \) large enough. This contradiction completes the lemma.

Based on above lemmas we obtain the main theorem of this paper which is stated as follows

**Theorem 2.5.** For each \( T > 0 \) eq. (1) has a unique solution on \([0, T]\).

**Proof.** First, from Lemma 2.3, there exists a finite random variable \( n_0 \) such that
\[ X^0_n(t) - X^0'_n(t) \geq \frac{1}{n_0} > 0 \] almost surely for any \( t \in [0,T] \). Therefore, the process
\[ X^n(t) = (X^n_1(t), X^n_2(t)) \] converges almost surely to a limit, called \( X(t) \) when \( n \) tends to infinity and \( X(t) \) satisfies eq. (1). This fact together with Lemma (2.4) leads to eq. (1) has a strong non-colliding solution.

Next, we show that eq. (1) has a unique solution in path-wise sense. Let \( X(t) \) and \( \overline{X}(t) \) be two solutions of eq. (1) on \([0, T]\). We have
\[ \left| X_1(t, \omega) - \overline{X}_1(t, \omega) \right| = \left| \int_0^t \left( \frac{\gamma}{X_1(s, \omega) - X_2(s, \omega)} + b_1(s, X(s, \omega)) - \frac{\gamma}{X_1(s, \omega) - \overline{X}_2(s, \omega)} - b_1(s, \overline{X}(s, \omega)) \right) ds \right| \]
\[ \leq \int_0^t \left( \frac{\gamma}{X_1(s, \omega) - X_2(s, \omega)} - \frac{\gamma}{X_1(s, \omega) - \overline{X}_2(s, \omega)} \right) ds + \int_0^t \left| b_1(s, X(s, \omega)) - b_1(s, \overline{X}(s, \omega)) \right| ds. \]  
(12)

Using the continuous property of the sample paths of \( X(t) \) and \( \overline{X}(t) \) and Lemma 2.4, we have
\[ m_0 = \min_{t \in [0,T]} \{ X_2(t, \omega) - X'_2(t, \omega), \overline{X}_2(t, \omega) - \overline{X}_1(t, \omega) \} > 0. \]

This fact together with the Lipschitz condition of \( b \) leads to
\[ \left| X_1(t, \omega) - \overline{X}_1(t, \omega) \right| \leq \int_0^t \frac{\gamma}{m_0^2} \left| X_2(s, \omega) - \overline{X}_2(s, \omega) \right| ds + \int_0^t \left| L X(s, \omega) - \overline{X}(s, \omega) \right| ds. \]  
(13)

Similarly, we estimate \( \left| X_2(t, \omega) - \overline{X}_2(t, \omega) \right| \). We obtain
\[
\sum_{i=1}^{2} \left| X_i(t, \omega) - \overline{X}_i(t, \omega) \right| \leq \left( \frac{2 \gamma}{m_0^2} + 2L \right) \int_0^t \sum_{i=1}^{2} \left| X_i(s, \omega) - \overline{X}_i(s, \omega) \right| ds.
\] (14)

It follows from Gronwall’s inequality that
\[
\sum_{i=1}^{2} \left| X_i(t, \omega) - \overline{X}_i(t, \omega) \right| = 0, \quad \text{for all } t \in [0, T].
\]

Therefore, \( X(t, \omega) = \overline{X}(t, \omega) \) for all \( t \in [0, T] \). The uniqueness has been concluded.

3. CONCLUSION

The main result proved in this paper is the existence and uniqueness of strong non-colliding solution in path-wise sense to the two-dimensional fractional non-colliding particle systems. From this result, we can propose a numerical approximation for this system.

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